J. Fluid Mech. (1975), *vol.* 71, *part* 4, *pp.* 625–673
Printed in Great Britain

Contributions to the theory of aerodynamic sound, with application to excess jet noise and the theory of the flute

By M. S. HOWE
Engineering Department, University of Cambridge

(Received 11 February 1975)

This paper describes a reformulation of the Lighthill (1952) theory of aerodynamic sound. A revised approach to the subject is necessary in order to unify the various *ad hoc* procedures which have been developed for dealing with aerodynamic noise problems since the original appearance of Lighthill's work. First, Powell's (1961*a*) concept of *vortex sound* is difficult to justify convincingly on the basis of Lighthill's acoustic analogy, although it is consistent with model problems which have been treated by the method of matched asymptotic expansions. Second, Candel (1972), Marble (1973) and Morfey (1973) have demonstrated the importance of *entropy* inhomogeneities, which generate sound when accelerated in a mean flow pressure gradient. This is arguably a more significant source of acoustic radiation in hot subsonic jets than pure jet noise. Third, the analysis of Ffowcs Williams & Howe (1975) of model problems involving the convection of an entropy 'slug' in an engine nozzle indicates that the whole question of *excess jet noise* may be intimately related to the convection of flow inhomogeneities through mean flow pressure gradients. Such problems are difficult to formulate precisely in terms of Lighthill's theory because of the presence of an extensive, non-acoustic, non-uniform mean flow. The convected-entropy source mechanism is actually *absent* from the alternative Phillips (1960) formulation of the aerodynamic sound problem.

In this paper the form of the acoustic *propagation operator* is established for a non-uniform mean flow in the absence of vortical or entropy-gradient source terms. The natural thermodynamic variable for dealing with such problems is the *stagnation enthalpy*. This provides a basis for a new acoustic analogy, and it is deduced that the corresponding acoustic source terms are associated solely with regions of the flow where the vorticity vector and entropy-gradient vector are non-vanishing. The theory is illustrated by detailed applications to problems which, in the appropriate limit, justify Powell's theory of vortex sound, and to the important question of noise generation during the unsteady convection of flow inhomogeneities in ducts and past rigid bodies in free space. At low Mach numbers wave propagation is described by a convected wave equation, for which powerful analytical techniques, discussed in the appendix, are available and are exploited.

Fluctuating heat sources are examined: a model problem is considered and provides a positive comparison with an alternative analysis undertaken elsewhere. The difficult question of the scattering of a plane sound wave by a

cylindrical vortex filament is also discussed, the effect of dissipation at the vortex core being taken into account.

Finally an approximate aerodynamic theory of the operation of musical instruments characterized by the flute is described. This involves an investigation of the properties of a vortex shedding mechanism which is coupled in a nonlinear manner to the acoustic oscillations within the instrument. The theory furnishes results which are consistent with the playing technique of the flautist and with simple acoustic measurements undertaken by the author.

CONTENTS

1. Introduction

Theories of aerodynamic sound

The theory of aerodynamically generated sound has been reviewed by Ffowcs Williams (1969), who detailed four essentially different and alternative significant theoretical developments which have occurred since the appearance of Lighthill's (1952) general theory. The first of these is due to Liepmann (1954, unpublished), who regarded the sound radiated from a turbulent region as driven by an ideal boundary which faithfully follows the profile of an instantaneous displacement thickness. Apart from the investigation reported in Laufer, Ffowcs Williams & Childress (1964), this approach has not been pursued to any great extent.

The second is that of Phillips (1960), who devised a theory of sound radiation from supersonic jets in which convection and refraction in temperature and velocity gradients produce substantial modifications of the radiation field. Lighthill (1952) had already analysed the effects of convection at *low* Mach numbers. Later Lilley (1958) had made the first serious attempt to extend Lighthill's theory to high-speed jets in an effort to explain such features as are evident in the experimental results of Atvars, Schubert & Ribner (1965). After the appearance of Phillips' work, Ffowcs Williams (1963) demonstrated that many of the important results involving high-speed convection could actually be deduced from Lighthill's acoustic analogy.

The third important development emerged through the attempts by various authors, in particular Crow (1970) and Lauvstad (1968), to formalize the theory of aerodynamic sound by means of the method of matched asymptotic expansions. Fluctuations within an acoustically compact region of turbulence scale on the eddy size l, say, whereas the appropriate scale in the far field is the acoustic wavelength $\lambda = O(l/M) \gg l$ for a sufficiently small turbulence Mach number M. The existence of two characteristic length scales in different regions of the flow implies that an asymptotic expansion of the disturbed flow in powers of the Mach number assumes essentially distinct forms in each of these regions, the forms being mutually consistent provided that certain matching conditions are satisfied in an overlap region. The major conclusion of this approach in the particular case of an isentropic inviscid fluid is that the leading term in the asymptotic expansion of the acoustic field generated by a compact region of turbulence arises from the Reynolds-stress contribution $\rho_0 v_i v_j$ to the Lighthill tensor (equation (1. 2) below), in which ρ_0 is the mean fluid density and $\mathbf{v}$ the turbulent velocity that would exist if the compressibility of the fluid were ignored.

Matched asymptotic expansions have been used to solve specific sound-generation problems by Müller & Obermeier (1967), Obermeier (1967), Rahman (1971), Crighton (1972) and Cannell & Ffowcs Williams (1973). The formal procedure, however, is not free from ambiguity, and some care must be exercised to ensure that misleading results are not obtained. The situation in this respect is actually more serious than might be concluded from an examination of the literature cited above. For example, at the time of writing, it is apparently not possible to treat the relatively simple problem of the scattering of a plane sound wave by a compact rigid body. Indeed an application of the general method described in the review article of Crighton & Lesser (1974) results in the prediction of the dipole component of the scattered field, but fails to predict the equally important monopole contribution, which is caused by the finite volume of fluid displaced by the body.

The final major alternative mentioned by Ffowcs Williams arises from the analysis of the energy balance equations of acoustics undertaken by Morfey (1966). However a significant degree of further development is still necessary if this approach is to provide predictions comparable with those of the Lighthill theory.

The acoustic wave operator

In Lighthill's (1952) fundamental paper on the acoustic analogy the momentum and continuity equations of fluid mechanics were combined to form

$$\Box^2\rho \equiv \frac{\partial^2\rho}{\partial t^2} - c^2\nabla^2\rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \tag{1.1}$$

where ρ is the fluid density and c the speed of sound in free space. The Lighthill stress tensor is defined by

$$T_{ij} = \rho v_i v_j + p_{ij} - c^2\rho\delta_{ij}, \tag{1.2}$$

p_{ij} being the compressive stress tensor.

Equation (1.1) is a nonlinear partial differential equation the exact solution of which can be obtained only when use is made of other equations of fluid mechanics which relate the pressure p, density ρ and the velocity v_i. But Lighthill argued that, since the propagation of small density perturbations in free space is governed by $\square^2\rho = 0$, the right-hand side of (1.1) may formally be regarded as a quadrupole distribution of sources which generate acoustic waves in an ideal fluid at rest. When T_{ij} is small except in an acoustically compact region of space, the Lighthill tensor specifies in an essentially unambiguous manner the actual source of sound. In more complicated and extensive flow regimes, however, a portion of the tensor must be responsible for the refraction and scattering of sound by flow inhomogeneities, and this can result in a significant modification of the acoustic field.

In order to avoid these difficulties Phillips (1960) derived the following *convected* wave equation for the logarithm of the pressure field:

$$\left\{\frac{D^2}{Dt^2}-\frac{\partial}{\partial x_j}\left(c^2\frac{\partial}{\partial x_j}\right)\right\}\ln\left(\frac{p}{p_0}\right) = \gamma\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}+\gamma\frac{D}{Dt}\left(\frac{1}{c_p}\frac{DS}{Dt}\right)-\gamma\frac{\partial}{\partial x_i}\left\{\frac{1}{\rho}\frac{\partial}{\partial x_j}[\mu\,(\epsilon_{ij}-\tfrac{2}{3}\theta\delta_{ij})]\right\}, \tag{1.3}$$

where p_0 is a reference pressure, S the specific entropy, $\gamma = c_p/c_v$ the ratio of the specific heat at constant pressure to that at constant volume, ϵ_{ij} the rate-of-strain tensor, θ the dilatation and μ is the viscosity coefficient of a Stokesian fluid. The fluid was assumed to satisfy the ideal-gas equation $p = \rho RT$, T being the temperature and R the gas constant. The material derivative D/Dt is equal to $\partial/\partial t + v_j\,\partial/\partial x_j$.

Phillips interpreted the terms on the right of (1.3) as acoustic sources. Thus the approach is based on the premise that the convected wave operator

$$\frac{D^2}{Dt^2}-\frac{\partial}{\partial x_j}\left(c^2\frac{\partial}{\partial x_j}\right)$$

correctly describes the propagation of small disturbances through a medium of variable mean properties. This must be the case at sufficiently high frequencies, but the assumption is without foundation at arbitrary acoustic wavelengths. Phillips' equation has also been applied to jet noise by Pao (1971, 1973), although in view of the doubtful validity of the wave operator the detailed conclusions of these analyses may require revision.

That the fundamental premise of the Phillips theory is suspect was recognized by Lilley (see, for example, Lilley 1973), who pointed out that in applications to shear-flow problems the source terms on the right of (1.3) actually contain contributions which are *linear* in the acoustic perturbation, and should preferably be included in the propagation operator. Lilley advanced the view that the correct wave operator is obtained when all linear terms in the perturbed flow are taken to the left-hand side of the acoustic-analogy equation, and in the particular case of slowly diverging shear flows derived a *third*-order differential equation for the pressure perturbation in which the source terms are nonlinear in the fluctuating variables. Unfortunately there exist eigensolutions of this equation which characterize the basic hydrodynamic instability of the shear layer, and the application of Lilley's equation to acoustic problems appears to involve an *ad hoc* decoupling of these instability modes from the acoustic field. The procedure is difficult to justify since the properties of the turbulent eddies responsible for the generation of sound are intimately related to the shear-layer instability. This difficulty does not arise with Phillips' equation (1.3), although we shall argue below that it provides a description of noise generation which is in some respects incorrect.

Doak (1973) has generalized the approach developed by Lilley and obtained a wave equation for that component of the momentum potential which he associates with the acoustic perturbations. However detailed practical conclusions may not be possible, and must await the further development of Doak's theory.

Vortex sound and entropy inhomogeneities

Powell (1961*a*, 1964) has proposed a theory of *vortex sound* in which the vorticity within a compact eddy in a weakly compressible, isentropic medium is identified as the basic source element in that it is considered to induce the whole flow, both the hydrodynamic turbulent field *and* the acoustic far field. This is an appealing point of view since it implies that acoustic sources are associated with regions of the flow in which the vorticity vector is non-vanishing, rather than with the more extensive hydrodynamic region which arises in Lighthill's acoustic-analogy theory. However, Powell's ideas have received little attention in the intervening period, although Lauvstad (1968, 1974) has, with some justification, cast doubts on the validity of the detailed analytical arguments used by Powell. Several model problems which have been treated by matched asymptotic expansions have actually been shown to be consistent with Powell's approach (e.g. Stüber 1970; Crighton 1972; Cannell & Ffowcs Williams 1973), which suggests that it ought to be possible to manipulate the equations of fluid mechanics in such a manner that Powell's source term would be seen to be dominant at low turbulence Mach numbers.

The recent calculations of Candel (1972) and Marble (1973) have illustrated that entropy inhomogeneities, produced by non-uniform combustion for example, in a jet flow are responsible for a significant proportion of the jet noise at subsonic Mach numbers. Morfey (1973) used Lighthill's theory to demonstrate that such sources arise through the interaction of the inhomogeneity with the mean flow pressure gradient, and Ffowcs Williams & Howe (1975) have analysed in detail the sound generated when an entropy 'slug' convects in a mean flow through a contraction in a duct or out of a nozzle. It is possible that this source mechanism plays a major role in the issue of *excess jet noise*, and in this respect it is significant that, in a situation where mean flow effects are crucial in determining the sound output, the mechanism is actually *absent* from Phillips' formulation of the jet noise problem.

The contribution of the present paper

The present paper constitutes an attempt to clarify certain aspects of the rather confused picture which emerges from the above summary. We shall do this by means of a reformulation of Lighthill's acoustic analogy which is capable of dealing unambiguously with situations in which there exists an extensive non-acoustic, non-uniform mean flow, for example near a bluff body in an air stream or in the region of a contraction in an engine duct. The starting point (§ 2) consists of a discussion of Crow's (1970) penetrating analysis of Lighthill's theory, from which it is concluded that the effective sound-producing region of a compact eddy is that in which the vorticity vector $\boldsymbol{\omega}$ is non-vanishing. In §§ 3 and 4 a modified wave equation valid for an ideal gas in arbitrary mean motion is obtained in which the fundamental acoustic variable is the *stagnation enthalpy*. This is the thermodynamic variable which emerges as the natural choice in the course of the analysis. The source terms for this variable are confined to regions of non-vanishing vorticity and entropy gradient.

The remaining sections of the paper are devoted to detailed applications of the new wave equation to specific acoustic problems. The range of these applications has been restricted to situations in which the mean flow is of small but *non-negligible* Mach number, the problems having been selected to demonstrate the power of the theory especially when used in conjunction with the method of low frequency Green's functions (Howe 1975) in order to deal with problems involving interactions with rigid bodies. An appendix has been included which contains a brief account of the relevant theory of Green's functions.

In § 5 the basic principles of vortex sound are discussed in terms of two canonical problems. The first is that of sound generation by a cylindrical vortex of elliptic cross-section, which may be regarded as the simplest model of a two-dimensional turbulent eddy. The second is an analysis of Crighton's (1972) problem of the sound generated when a line vortex negotiates a path around the edge of a rigid semi-infinite plane. Here the theory reveals that the instantaneous intensity of the acoustic radiation is determined by the rate at which the line vortex cuts across a hypothetical field of streamlines describing potential flow about the half-plane. More general problems associated with the convection of turbulent eddies and entropy inhomogeneities through variable-geometry ducts and past bluff bodies in free space are then discussed (§§ 6 and 7). These are of considerable interest in connexion with the theory of excess jet noise. In § 8 the question of the generation of sound by fluctuating heat sources is examined in an attempt to assess the validity of the entropy source terms of the theory, predictions for a model problem being shown to be consistent with calculations undertaken elsewhere. The difficult question of the scattering of a plane sound wave by a line vortex is taken up in § 9, where a circular cylindrical vortex is considered and account is taken of the entropy variation in the vortex core arising from viscous dissipation within the region of large velocity gradients.

In the final section of the paper (§ 10) we present a detailed aerodynamic theory of the operation of a musical instrument such as the flute. A vortex shedding model of the acoustic source mechanism is examined. The intensity and properties of the vorticity are coupled to the flautist's blowing pressure and to the acoustic cross-flow velocity at the mouth of the flute, and are determined by means of a nonlinear consistency condition during the course of the analysis. The model, though undoubtedly crude, apparently explains several of the practical techniques employed by the flautist in playing his instrument. The discussion of this section demonstrates the power and elegance achieved by conducting the analysis in terms of the stagnation enthalpy variable and the appropriate low frequency Green's function.

2. Sound radiation by a compact turbulent eddy

Crow (1970) has considered the radiation of sound by a compact vortical eddy region of low Mach number M and of typical length scale l. The wavelength of the sound generated $\lambda = O(l/M)$ greatly exceeds the dimensions of the eddy, and the question of describing the flow in various parts of the fluid can then be posed in terms of a singular perturbation problem in which quantities in the eddy region are scaled on the length l and the time l/u (u being the root-mean-square turbulent velocity), the appropriate length scale in the acoustic region being λ.

By this means Crow demonstrated that the leading term in the asymptotic expansion for the acoustic field of a compact eddy located in an isentropic medium at rest is obtained by approximating the Lighthill tensor by

$$T_{ij} = \rho_0 v_i v_j, \tag{2.1}$$

where ρ_0 is the undisturbed, constant mean density of the fluid. The velocity $\mathbf{v}$ is the divergence-free, vortically generated fluid velocity defined by means of the following argument.

Consider an incompressible inviscid flow characterized by the vorticity distribution $\boldsymbol{\omega}$. Since the resulting velocity is divergence free it is entirely determined by $\boldsymbol{\omega}$, and is actually given in terms of the vector potential $\mathbf{A}$ by

$$\mathbf{v} = \operatorname{curl} \mathbf{A}, \quad \mathbf{A} = \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d^3\mathbf{y}. \tag{2.2}$$

It is a straightforward matter to deduce that $\mathbf{v} = O(m^*/|\mathbf{x}|^3)$ as $|\mathbf{x}| \to \infty$, provided that the integral

$$\mathbf{m}^* = \int \mathbf{y} \wedge \boldsymbol{\omega}\, d^3\mathbf{y} \tag{2.3}$$

converges. But for an initially bounded region of vorticity in an isentropic medium, Kelvin's theorem, which asserts that vortex lines move with the fluid particles, ensures that the vortical region will be bounded at any subsequent time, and that the integral in (2.3) is always well defined. This is a purely kinematic result which does not depend on the initial hypothesis that the flow be incompressible.

Thus in the aerodynamic sound problem Crow defined the total fluid velocity $\mathbf{u}$ by

$$\mathbf{u} = \mathbf{v} + \nabla\psi, \tag{2.4}$$

where $\mathbf{v}$ is given in terms of the vorticity by (2.2). Thus $\operatorname{div} \mathbf{v} = 0$. The potential function ψ is determined in terms of the density fluctuations of the medium by means of the continuity equation

$$\rho^{-1} D\rho/Dt + \nabla^2\psi = 0. \tag{2.5}$$

Now in the region of the eddy flow

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\rho c^2}\frac{Dp}{Dt} = O\left(\frac{u}{l}M^2\right),$$

so that (2.5) implies that

$$\psi = O(luM^2) \tag{2.6}$$

in the eddy region.

Next consider the implications of Crow's result (2.1). Inserting this into Lighthill's equation (1.1), we obtain for the acoustic perturbation density

$$\frac{\rho}{\rho_0} = \frac{1}{4\pi c^2|\mathbf{x}|}\int \frac{\partial^2 v_i v_j}{\partial y_i \partial y_j}\delta\left\{t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c}\right\} d^3\mathbf{y}\, d\tau, \tag{2.7}$$

where use has been made of the free-space Green's function given in equation (A 4) of the appendix. Using the identity

$$\partial^2 v_i v_j/\partial x_i \partial x_j = \operatorname{div}(\boldsymbol{\omega}\wedge\mathbf{v}) + \nabla^2(\tfrac{1}{2}v^2) \tag{2.8}$$

this becomes, in the usual manner,

$$\frac{\rho}{\rho_0} \simeq \frac{1}{4\pi c^2|\mathbf{x}|}\left\{\frac{-\mathbf{x}}{c|\mathbf{x}|}\cdot\frac{\partial}{\partial t}\int(\boldsymbol{\omega}\wedge\mathbf{v})\left(t-\frac{|\mathbf{x}-\mathbf{y}|}{c},\mathbf{y}\right)d^3\mathbf{y} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int v^2\left(t-\frac{|\mathbf{x}-\mathbf{y}|}{c},\mathbf{y}\right)d^3\mathbf{y}\right\}, \tag{2.9}$$

provided that $|\mathbf{x}| \gg l$.

Consider first the second integral in this result, I_2, say. In the leading approximation we may neglect retarded-time variations over the region where v^2 is significant, provided that this does not prejudice the convergence of the integral. But $v(t,\mathbf{y}) = O(m^*/|\mathbf{y}|^3)$ for large $|\mathbf{y}|$, so that no convergence difficulties will arise. Thus we have

$$I_2 \simeq \frac{1}{4\pi c^4|\mathbf{x}|}\frac{\partial^2}{\partial t^2}\int v^2(t-|\mathbf{x}|/c,\mathbf{y})\,d^3\mathbf{y} \tag{2.10}$$

for $|\mathbf{x}| \gg l$.

The order of magnitude of this expression can be estimated as follows. Using the velocity representation (2.4), the inviscid isentropic momentum equation can be expressed in the form

$$\frac{\partial\mathbf{v}}{\partial t} + \operatorname{grad}\left\{\int\frac{dp}{\rho} + \tfrac{1}{2}u^2 + \frac{\partial\psi}{\partial t}\right\} = -\boldsymbol{\omega}\wedge\mathbf{v} - \boldsymbol{\omega}\wedge\nabla\psi. \tag{2.11}$$

Take the scalar product of this equation with $\mathbf{v}$, and recall that $\operatorname{div}\mathbf{v} = 0$ to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}v^2 + \operatorname{div}\left\{\mathbf{v}\left[\int\frac{dp}{\rho} + \tfrac{1}{2}u^2 + \frac{\partial\psi}{\partial t}\right]\right\} = -\mathbf{v}\cdot\boldsymbol{\omega}\wedge\nabla\psi. \tag{2.12}$$

Integrating over all space and applying the divergence theorem, the contribution from the second term on the left vanishes because

$$\mathbf{v}\left\{\int\frac{dp}{\rho} + \tfrac{1}{2}u^2 + \frac{\partial\psi}{\partial t}\right\}$$

tends to zero at least as fast as $|\mathbf{y}|^{-3}$ as $|\mathbf{y}| \to \infty$. Hence

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\int v^2 d^3\mathbf{y} = -\frac{\partial}{\partial t}\int \mathbf{v} . \boldsymbol{\omega} \wedge \nabla\psi \, d^3\mathbf{y}. \tag{2.13}$$

The integration on the right is confined to the bounded region of the flow in which the vorticity is non-vanishing, so that its contribution may be estimated as being of order

$$\left(\frac{u}{l}\right)\left(\frac{u^2}{l}\right)(uM^2)\, l^3 = M^2u^4l,$$

use having been made of (2.6). Hence

$$I_2 = O\left(\frac{l}{|\mathbf{x}|}M^6\right). \tag{2.14}$$

Considering next the vortical integral I_1, say, in (2.9), the leading approximation here, viz.

$$-\frac{1}{4\pi c^3|\mathbf{x}|}\cdot\frac{\mathbf{x}}{|\mathbf{x}|}\cdot\frac{\partial}{\partial t}\int \boldsymbol{\omega} \wedge \mathbf{v}(t-|\mathbf{x}|/c, \mathbf{y})\, d^3\mathbf{y},$$

vanishes identically because the integrand can be expressed through (2.8) as a divergence. Higher-order approximations are obtained by expanding the integrand in powers of the retarded-time element $\mathbf{x}.\mathbf{y}/c|\mathbf{x}|$, a procedure which is valid provided that the vortical region is compact. The first term in such an expansion gives

$$I_1 \simeq \frac{1}{4\pi c^4|\mathbf{x}|}\frac{\partial^2}{\partial t^2}\int\left(\frac{\mathbf{x}.\mathbf{y}}{|\mathbf{x}|}\right)\left(\frac{\mathbf{x}.\boldsymbol{\omega}\wedge\mathbf{v}}{|\mathbf{x}|}\right)(t-|\mathbf{x}|/c, \mathbf{y})\, d^3\mathbf{y}, \tag{2.15}$$

from which we have

$$I_1 = O\left(\frac{l}{|\mathbf{x}|}M^4\right). \tag{2.16}$$

Thus $I_1 \gg I_2$ for small turbulence Mach numbers M. Using the estimate (2.16) in (2.9) leads directly to Lighthill's (1952) u^8 law, which is seen to depend for its validity on the non-vanishing of the moment integral in (2.15).

The above discussion has thus led us to the view, first proposed by Powell (1961 *a*), that the dynamical source of sound in low Mach number turbulence can be identified precisely with those regions of the flow in which the vorticity vector is non-vanishing. Explicitly, it appears that in the far field density perturbations can be calculated by means of the acoustic analogy embodied in the equation

$$\partial^2\rho/\partial t^2 - c^2\nabla^2\rho = \rho_0 \operatorname{div}(\boldsymbol{\omega}\wedge\mathbf{v}), \tag{2.17}$$

where the term on the right-hand side is calculated on the assumption that the flow is incompressible.

Actually this conclusion was foreshadowed in Lighthill's second fundamental paper (Lighthill 1954), in which it was pointed out that in subsonic jet flows the terms omitted from T_{ij} in arriving at the approximation in (2.17) are equivalent to an octupole and a term proportional to $c^{-4}\,\partial(p \operatorname{div} \mathbf{v})/\partial t$. The acoustic contribution of the latter is of order $(\sigma/l^3)(l/|\mathbf{x}|)M^6$, where σ is the volume of space in which the source is significant. The above discussion reveals that σ is essentially restricted to the region of non-vanishing vorticity.

Equation (2.17) has been derived for the rather special case of an isolated vortical region located in free space. It would be surprising if this conclusion were materially altered by the presence in the flow of scattering surfaces, although it is not immediately apparent how the more general result can be established on the basis of Lighthill's acoustic analogy. The analyses of §§ 3 and 4 will attempt to resolve this issue by means of a reformulation of the acoustic analogy in which the *stagnation enthalpy*, rather than the density, assumes the role of the fundamental acoustic variable.

3. Propagation of sound in an irrotational mean flow

The aerodynamic theory of sound associates the generation of noise with the presence of random distributions of vorticity and entropy inhomogeneities within a flow. In this section we shall be concerned with establishing the *propagation* properties of the sound. Therefore we shall initially assume that vorticity and entropy inhomogeneities are absent from the mean flow, and this implies, also, that viscosity and heat-conduction effects can be ignored.

Suppose that in the presence of an arbitrary distribution of scattering bodies the fluid is in a state of compressible *irrotational* mean flow specified by the velocity potential $\phi_0(\mathbf{x})$. Suppose further that an irrotational disturbance is introduced into the flow by the application of an impulsive force or otherwise, and let the perturbation in the potential be denoted by $\phi_1(\mathbf{x},t)$. Set

$$\phi(\mathbf{x},t) = \phi_0(\mathbf{x})+\phi_1(\mathbf{x},t). \tag{3.1}$$

We shall derive the equation satisfied by $\phi(\mathbf{x},t)$.

The equation of conservation of momentum in an irrotational isentropic flow possesses a first integral of the usual form

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\frac{\partial\phi}{\partial x_i}\right)^2+\int\frac{dp}{\rho} = \text{constant}. \tag{3.2}$$

The equation of continuity of mass is

$$\left.\begin{aligned} \rho^{-1}D\rho/Dt+\nabla^2\phi &= 0,\\ \text{where}\qquad \frac{D}{Dt} &= \frac{\partial}{\partial t}+\frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_j}. \end{aligned}\right\} \tag{3.3}$$

Since viscosity and heat conduction are negligible, the density and pressure variations within a fluid particle are related by

$$Dp/Dt = c^2D\rho/Dt, \tag{3.4}$$

where c is the speed of sound.

Equations (3.2) and (3.4) imply that

$$\frac{D}{Dt}\left\{\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\frac{\partial\phi}{\partial x_i}\right)^2\right\}+\frac{c^2}{\rho}\frac{D\rho}{Dt} = 0. \tag{3.5}$$

Hence, eliminating $\rho^{-1}D\rho/Dt$ between (3.3) and (3.5), we obtain

$$\left(\frac{\partial}{\partial t}+\frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_j}\right)\left(\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_i}\right)\phi - c^2\nabla^2\phi = 0. \tag{3.6}$$

Within the restrictions of the present hypotheses this equation is exact and nonlinear. But the potential function ϕ defined in (3.1) contains both the steady component of the velocity field and the perturbation field. The distinguishing feature of the two components is that only the perturbation potential is time dependent. In order to emphasize this time dependence it is appropriate to take the partial time derivative of (3.6). This has the advantage that in many situations it is legitimate to assume that $\dot\phi \equiv \partial\phi/\partial t$ is small.

In differentiating (3.6) partially with respect to time we shall, in order to avoid complications which are examined in detail in the next section, suppose that the speed of sound is time independent. Then it is a simple matter to deduce that

$$\frac{\partial}{\partial t}\left\{\left(\frac{\partial}{\partial t}+\frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_j}\right)\left(\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_i}\right)\phi\right\} \equiv \left\{\frac{D^2}{Dt^2}+\frac{D\mathbf{v}}{Dt}\cdot\frac{\partial}{\partial\mathbf{x}}\right\}\dot\phi, \tag{3.7}$$

where $D/Dt = \partial/\partial t+\mathbf{v}.\partial/\partial\mathbf{x}$ and $\mathbf{v} = \partial\phi/\partial\mathbf{x} \equiv \partial\phi_0/\partial\mathbf{x}+\partial\phi_1/\partial\mathbf{x}$. Hence it follows that the equation satisfied by $\dot\phi$ can be set in the form

$$\left\{\frac{D^2}{Dt^2}+\frac{D\mathbf{v}}{Dt}\cdot\frac{\partial}{\partial\mathbf{x}}-c^2\nabla^2\right\}\dot\phi = 0. \tag{3.8}$$

If the steady mean flow specified by $\phi_0(\mathbf{x})$ is known, then (3.8) constitutes a nonlinear equation describing the propagation through the compressible medium of time-dependent perturbations. The terms which are nonlinear in ϕ_1 describe the self-modulation of the perturbation wave field. In the *acoustic* approximation (3.8) can be linearized with respect to the perturbation potential, and since this is the only time-dependent component of the flow this is equivalent to neglecting ϕ_1 wherever it appears in the propagation operator. Let $\mathbf{U} = \nabla\phi_0$, then in this case we have

$$\left\{\left(\frac{\partial}{\partial t}+\mathbf{U}.\frac{\partial}{\partial\mathbf{x}}\right)^2+\frac{\partial}{\partial\mathbf{x}}(\tfrac{1}{2}U^2).\frac{\partial}{\partial\mathbf{x}}-c^2\nabla^2\right\}\dot\phi = 0. \tag{3.9}$$

In particular, if $U^2 \ll c^2$, the second term on the left of this result may be dropped, and $\dot\phi$ then satisfies the *convected* wave equation

$$\left(\frac{\partial}{\partial t}+\mathbf{U}.\frac{\partial}{\partial\mathbf{x}}\right)^2\dot\phi - c^2\nabla^2\dot\phi = 0. \tag{3.10}$$

This is the form used by Howe (1975) and by Ffowcs Williams & Howe (1975) in their treatments of the generation of sound by convected flow inhomogeneities. At sufficiently small mean flow Mach numbers the irrotational velocity field $\mathbf{U}$ is essentially divergence free, and solutions of (3.10) satisfy a reverse-flow reciprocal theorem (see appendix), which leads to a significant simplification of the analysis of certain flow–surface interaction problems. In this respect the perturbation potential is the *natural* variable with which to conduct the analysis, since the pressure and density perturbations do *not* satisfy the classical convected wave equation, even at low mean flow Mach numbers.

4. A reformulation of the acoustic analogy

The analysis of the previous section indicates that it might be appropriate to express the wave equation for acoustic disturbances in an *arbitrary* mean flow in a form which is similar to (3.8). Unfortunately the procedure adopted there involved the use of the Bernoulli integral (3.2), which is valid only for points of the flow exterior to entropy and vorticity inhomogeneities. The difficulty in the present case arises because it is no longer possible to assert that perturbations in the fluid motion can be described by means of a scalar potential ϕ alone.

Actually it was more convenient to employ $\dot{\phi}$, rather than ϕ, as the perturbation quantity in § 3, since this emphasizes the temporal variations characteristic of an acoustic field. The Bernoulli integral (3.2) reveals that this is equivalent to adopting

$$B = \int \frac{dp}{\rho} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x_i}\right)^2 \tag{4.1}$$

as the acoustic variable. In the absence of sound B is constant throughout the mean irrotational flow. It may be identified with the *stagnation enthalpy* per unit mass of an isentropic fluid.

In an arbitrary flow field the density ρ cannot normally be specified as a function of the pressure alone, so that a generalization of (3.8) will necessarily involve a widening of the definition of B. This can be done by introducing the specific *heat function* w (Landau & Lifshitz 1959, chap. 1), which is related to the pressure, density, temperature and specific entropy introduced in § 1 by the differential equality

$$dw = \rho^{-1}\, dp + T\, dS. \tag{4.2}$$

The generalization of (4.1) to all points of the flow is therefore effected by means of the definition

$$B(\mathbf{x}, t) = w + \tfrac{1}{2}v^2, \tag{4.3}$$

which is the thermodynamic potential describing the *specific stagnation enthalpy* or heat function.

In the following discussion we shall assume that it is legitimate to neglect the effects of viscous dissipation and heat conduction, although the latter restriction will subsequently be removed. Then at points exterior to flow inhomogeneities and where the mean temperature of the medium is uniform B must satisfy (3.8). With these restrictions in mind we now proceed to derive the general equation determining the stagnation enthalpy B.

First express the momentum equation in Crocco's form

$$\partial \mathbf{v}/\partial t + \operatorname{grad} B = -\boldsymbol{\omega} \wedge \mathbf{v} + T \operatorname{grad} S, \tag{4.4}$$

where $\mathbf{v}$ now denotes the *total* fluid velocity (cf. Liepmann & Roshko 1957, p. 193). Since the entropy of a fluid particle is conserved in the absence of viscous dissipation and heat conduction, the equation of continuity becomes

$$(\rho c^2)^{-1}\, Dp/Dt + \operatorname{div} \mathbf{v} = 0. \tag{4.5}$$

Take the divergence of (4.4) and the partial time derivative of (4.5) and subtract:

$$\frac{\partial}{\partial t}\left\{\frac{1}{\rho c^2}\frac{Dp}{Dt}\right\} - \nabla^2 B = \operatorname{div}\{\boldsymbol{\omega}\wedge\mathbf{v} - T\operatorname{grad}S\}. \tag{4.6}$$

Next we tentatively write this in the form

$$\frac{1}{c^2}\frac{D^2B}{Dt^2} + \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot\frac{\partial B}{\partial \mathbf{x}} - \nabla^2 B = \operatorname{div}\{\boldsymbol{\omega}\wedge\mathbf{v} - T\operatorname{grad}S\}$$
$$+\left\{\frac{1}{c^2}\frac{D^2B}{Dt^2} + \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot\frac{\partial B}{\partial \mathbf{x}} - \frac{\partial}{\partial t}\left(\frac{1}{\rho c^2}\frac{Dp}{Dt}\right)\right\}. \tag{4.7}$$

The left-hand side of this equation resembles the left-hand side of (3.8), but it must be borne in mind that it may be necessary to augment the left side of (4.7) with terms linear in B which arise because of variations in the speed of sound [assumed constant in the derivation of (3.8)].

The problem now consists of attempting to simplify the last term on the right of (4.7). In doing this we shall assume further that the fluid satisfies the ideal-gas equation

$$p = \rho RT. \tag{4.8}$$

First observe that from (4.4)

$$\frac{D\mathbf{v}}{Dt}\cdot\frac{\partial B}{\partial \mathbf{x}} = -\frac{\partial \mathbf{v}}{\partial t}\cdot\frac{D\mathbf{v}}{Dt} - \frac{D\mathbf{v}}{Dt}\cdot[\boldsymbol{\omega}\wedge\mathbf{v} - T\operatorname{grad}S], \tag{4.9}$$

and substitute this result into the right-hand side of (4.7). The remaining terms involving $\mathbf{v}$, B and p can then be expressed in terms of the derivatives of p alone, using (4.8), the entropy equation $DS/Dt = 0$ and the momentum and continuity equations. In this way the right-hand side of (4.7) is eventually reduced to

$$\operatorname{div}\{\boldsymbol{\omega}\wedge\mathbf{v} - T\operatorname{grad}S\} - \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot[\boldsymbol{\omega}\wedge\mathbf{v} - T\operatorname{grad}S] - \frac{1}{\rho}\frac{\partial p}{\partial t}\cdot\frac{D}{Dt}\left(\frac{1}{c^2}\right).$$

Now

$$\frac{1}{\rho}\frac{\partial p}{\partial t} = \frac{1}{\rho}\frac{Dp}{Dt} - \frac{v_j}{\rho}\frac{\partial p}{\partial x_j}$$
$$= \frac{Dw}{Dt} + \frac{D}{Dt}(\tfrac{1}{2}.\mathbf{v}^2) \equiv \frac{DB}{Dt}, \tag{4.10}$$

and hence

$$\frac{1}{\rho}\frac{\partial p}{\partial t}\frac{D}{Dt}\left(\frac{1}{c^2}\right) = \frac{D}{Dt}\left(\frac{1}{c^2}\right)\frac{DB}{Dt}, \tag{4.11}$$

a term linear in B which arises because of the variation in the speed of sound in the fluid. It should properly be incorporated into the wave operator on the left of (4.7).

Taking all of these points into consideration we finally reduce (4.7) to the form

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right) + \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot\frac{\partial}{\partial \mathbf{x}} - \nabla^2\right\}B$$
$$= \operatorname{div}\{\boldsymbol{\omega}\wedge\mathbf{v} - T\operatorname{grad}S\} - \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot[\boldsymbol{\omega}\wedge\mathbf{v} - T\operatorname{grad}S]. \tag{4.12}$$

Within the confines of an inviscid non-conducting ideal-gas theory, this equation is *exact*. Note that the material derivative D/Dt involves the *actual* fluid velocity and not merely that of a mean flow.

In certain applications to be described below we shall be particularly interested in situations in which it is not possible to leave out the effects of heat conduction. In this case DS/Dt no longer vanishes, and the equation of continuity assumes the form

$$\frac{1}{\rho c^2}\frac{Dp}{Dt}+\operatorname{div}\mathbf{v} = \frac{1}{c_p}\frac{DS}{Dt}, \tag{4.13}$$

where c_p is the specific heat at constant pressure. Proceeding as in the above analysis, we deduce that the equation which relates the variations in the stagnation enthalpy to the vorticity $\boldsymbol{\omega}$ and specific entropy S is now

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right)+\frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot\frac{\partial}{\partial\mathbf{x}}-\nabla^2\right\}B$$
$$= \operatorname{div}\{\boldsymbol{\omega}\wedge\mathbf{v}-T\operatorname{grad}S\}-\frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot\{\boldsymbol{\omega}\wedge\mathbf{v}-T\operatorname{grad}S\}+\frac{D}{Dt}\left(\frac{T}{c^2}\frac{DS}{Dt}\right)+\frac{\partial}{\partial t}\left(\frac{1}{c_p}\frac{DS}{Dt}\right). \tag{4.14}$$

At points of the flow exterior to vorticity and entropy inhomogeneities, the terms on the right of (4.12) and (4.14) vanish identically, and the irrotational perturbation flow equation (3.8) is obtained, but with account taken of the variations in the sound speed. The linearized form of that equation describes the propagation of small acoustic disturbances in the mean irrotational flow. The Lighthill (1952) acoustic analogy is based on just such an identification of part of the general Navier–Stokes equation with the wave operator $\Box^2$ in space devoid of vorticity and entropy fluctuations. It is natural therefore to pursue such an analogy in the present case. The terms on the right-hand sides of (4.12) and (4.14) then assume the roles of *inhomogeneous acoustic source terms*, but they have the distinctive property of being confined solely to regions of the flow where the vorticity and entropy-gradient vectors are non-vanishing. When the characteristic Mach number is sufficiently small and the flow is isentropic, the second term on the right of (4.12) may be neglected and, in the absence of a mean flow, that equation then reduces essentially to Powell's result embodied in (2.17).

Boundary conditions and energy flux

Many of the applications of (4.12) and (4.14) described in the following sections will involve the interaction of a flow field with a rigid surface. In most cases the inhomogeneous source terms will be localized in free space, so that in the immediate vicinity of the surface Crocco's equation (4.4) becomes

$$\partial\mathbf{v}/\partial t+\operatorname{grad}B = 0. \tag{4.15}$$

It follows that on the rigid surface the normal derivative $\partial B/\partial n$ of the stagnation enthalpy must vanish. The one exception to this rule which we shall examine is considered in § 8.

Let us also note that the energy equation for an ideal, inviscid, non-heat-conducting fluid has the form

$$\partial E/\partial t + \operatorname{div}[\rho \mathbf{v} B] = 0 \tag{4.16}$$

(Landau & Lifshitz 1959, p. 12), where

$$E = \tfrac{1}{2}\rho v^2 + \rho\epsilon$$
$$= \text{kinetic energy per unit volume} + \text{internal energy per unit volume.}$$

Thus in particular applications the acoustic energy flux can be obtained from the general energy flux vector $\rho\mathbf{v}B$.

5. The principles of vortex sound

In this section we shall apply (4.12) to solve two relatively elementary problems of acoustics, one of which has been examined elsewhere. The first is that of sound generation by a cylindrical vortex of elliptic cross-section in a weakly compressible fluid. This is perhaps a more realistic model of a two-dimensional eddy than that which consists of the pair of spinning vortices treated by Obermeier (1967) and Stüber (1970). The second problem is that already solved by Crighton (1972) by matched asymptotic expansions, and involves the determination of the sound radiated during the passage of a line vortex around the edge of a rigid half-plane. We shall examine this with the aid of the theory of low frequency Green's functions discussed in the appendix. This method has the distinct advantage over the formal matching procedure of enabling the solution to be cast naturally into a physically meaningful mathematical form which gives a general indication of the mechanism associated with the generation of sound by turbulence located in the vicinity of a rigid body. In both of these applications we shall be concerned with low Mach number disturbances in an isentropic fluid. In this case we need retain only the Powell dipole source $\operatorname{div}(\boldsymbol{\omega}\wedge\mathbf{v})$ on the right of (4.12).

Radiation of sound by an elliptic vortex

Consider a cylindrical region of elliptic cross-section within which the fluid has uniform vorticity Ω per unit area in a direction parallel to the axis of the cylinder. The latter is assumed to lie along the $i = 3$ axis of a rectangular co-ordinate system ($i = 1, 2, 3$) in which $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ denote unit vectors parallel respectively to each of these axes. The ellipse is only slightly deformed from a circle of radius a, with semi-major and semi-minor axes respectively equal to $a(1\pm\epsilon)$ $(0 < \epsilon \ll 1)$.

If $a\Omega/c$ is sufficiently small the flow in the region of the vortex is essentially incompressible and the polar equation of the ellipse can be expressed in the form (Lamb 1932, p. 231)

$$R = a\{1 + \epsilon\cos[2\theta - \tfrac{1}{2}\Omega t]\}, \tag{5.1}$$

where $R = (x_1^2 + x_2^2)^{\frac{1}{2}}$. When $\epsilon = 0$ the flow exterior to the vortex is steady (B = constant) and no sound is emitted. When ϵ is small but finite, however, the elliptic cross-section of the cylinder spins about its axis with angular velocity $\frac{1}{4}\Omega$, and the unsteadiness induced in the flow radiates as sound.

We now linearize the wave operator on the left of (4.12) about the mean flow. At the same time the material derivative D/Dt may be approximated by the partial derivative $\partial/\partial t$ provided that $U/c \ll 1$, where $U = \frac{1}{2}a\Omega$ is the maximum value of the mean flow speed, which occurs at the boundary of the vortex core. Hence (4.12) reduces to

$$\{c^{-2}\,\partial^2/\partial t^2 - \nabla^2\}B = \operatorname{div}(\boldsymbol{\omega}\wedge\mathbf{v}). \tag{5.2}$$

Within the vortex core the vorticity is given by

$$\boldsymbol{\omega} = \Omega\mathbf{k} \tag{5.3}$$

and the flow velocity by

$$\mathbf{v} = -\tfrac{1}{2}\Omega R[\sin\theta + \epsilon\sin(\theta-\tfrac{1}{2}\Omega t)]\,\mathbf{i} + \tfrac{1}{2}\Omega R[\cos\theta - \epsilon\cos(\theta-\tfrac{1}{2}\Omega t)]\,\mathbf{j} \tag{5.4}$$

(Lamb 1932, p. 231), so that

$$\boldsymbol{\omega}\wedge\mathbf{v} = -\tfrac{1}{2}\Omega^2\mathbf{x} + \tfrac{1}{2}\epsilon\Omega^2 R[\mathbf{i}\cos(\theta-\tfrac{1}{2}\Omega t) - \mathbf{j}\sin(\theta-\tfrac{1}{2}\Omega t)], \tag{5.5}$$

where $\mathbf{x} = (x_1, x_2)$.

To simplify the details of the analysis we first determine the effective multipole strength of the acoustic source term $\operatorname{div}(\boldsymbol{\omega}\wedge\mathbf{v})$ by multiplying the right-hand side of (5.2) by a 'test' function $f(\mathbf{x})$, say, and integrating over a typical cross-section of the vortex. Thus we consider

$$\begin{aligned} I &= \int \operatorname{div}(\boldsymbol{\omega}\wedge\mathbf{v})f(\mathbf{x})\,d^2\mathbf{x} \\ &= -\int \boldsymbol{\omega}\wedge\mathbf{v}\,.\,\nabla f\,d^2\mathbf{x} \\ &= -\int(\boldsymbol{\omega}\wedge\mathbf{v})_i\left\{\frac{\partial f}{\partial x_i} + x_j\frac{\partial^2 f}{\partial x_i\,\partial x_j} + \ldots\right\}d^2\mathbf{x}, \end{aligned} \tag{5.6}$$

the expansion in curly brackets being about the origin $(x_1, x_2) = 0$.

Using (5.1) and (5.5) the leading time-dependent term in this expansion is found to be given by

$$I = T_{ij}\frac{\partial^2 f}{\partial x_i\,\partial x_j}, \quad \text{where} \quad \mathsf{T} = \tfrac{1}{8}\epsilon\pi\Omega^2 a^4\begin{pmatrix}\cos\frac{1}{2}\Omega t & \sin\frac{1}{2}\Omega t\\ \sin\frac{1}{2}\Omega t & -\cos\frac{1}{2}\Omega t\end{pmatrix}, \tag{5.7}$$

with i and j ranging over the 1 and 2 directions only.

It follows from this result that the time-dependent part of the acoustic source term is given in the leading approximation by the line quadrupole distribution

$$\operatorname{div}(\boldsymbol{\omega}\wedge\mathbf{v}) \simeq \partial^2\{T_{ij}\,\delta(x_1)\,\delta(x_2)\}/\partial x_i\,\partial x_j. \tag{5.8}$$

The acoustic field is now obtained by convoluting this result with the free-space Green's function associated with the wave operator on the left of (5.2) (see equation (A 4) of the appendix). Introducing the vortex core strength $\kappa = \frac{1}{2}\Omega a^2$ we obtain

$$\begin{aligned} B &= \frac{1}{4\pi}\int\frac{\partial^2}{\partial y_i\,\partial y_j}\{T_{ij}(\tau)\,\delta(y_1)\,\delta(y_2)\}\frac{\delta\{t-\tau-|\mathbf{x}-\mathbf{y}|/c\}}{|\mathbf{x}-\mathbf{y}|}\,d^3\mathbf{y}\,d\tau \\ &= -\frac{\epsilon\kappa^2\Omega^2}{32c^2}\int_{-\infty}^{\infty}\frac{R^2}{(R^2+y_3^2)^{\frac{3}{2}}}\cos\left\{2\theta - \frac{\Omega t}{2} + \frac{\Omega}{2c}(R^2+y_3^2)^{\frac{1}{2}}\right\}dy_3, \end{aligned} \tag{5.9}$$

where $(x_1, x_2) = R(\cos\theta, \sin\theta)$.

In the acoustic far field, $\Omega R/c \gg 1$, this integral can be evaluated by the method of stationary phase. Noting also that in this problem the acoustic pressure perturbation p is related to the acoustic component of the stagnation enthalpy by $p = \rho_0 B$, we finally deduce that

$$p \simeq -\frac{\epsilon(2\pi)^{\frac{1}{2}}}{8}\rho_0 U^2 M^{\frac{3}{2}}\left(\frac{a}{R}\right)^{\frac{1}{2}}\cos\left\{2\theta+\frac{\pi}{4}-\frac{\Omega}{2}\left(t-\frac{R}{c}\right)\right\}, \tag{5.10}$$

where $M = U/c$ and $\Omega R/c \gg 1$.

This is the expected form of the radiation field of a two-dimensional eddy (cf. Ffowcs Williams 1969), in which the perturbation pressure decays inversely as the square root of the distance R, and in which the mean-square pressure level has a characteristic U^4M^3 parametric dependence on the eddy velocity. The acoustic frequency is twice the rotation frequency of the elliptic vortex core, and the instantaneous directivity has the quadrupolar $\cos 2\theta$ dependence on the angle.

Radiation from vortex interaction with a rigid half-plane

The two-dimensional example treated above illustrates the manner in which the dipole source $\operatorname{div}(\boldsymbol{\omega} \wedge \mathbf{v})$ in free space is actually acoustically equivalent to a quadrupole distribution, in accordance with Lighthill's (1952) theory and the discussion of § 2. The situation is generally different, however, when the vortical region is located in the vicinity of a rigid body. Now the turbulent velocity fluctuations exert an unsteady force distribution on the body whose instantaneous resultant integrates to zero only in special circumstances, exhibited, for example, by an infinitely extended flat plate. In this case the scattered sound field is typically of the more powerful dipole type considered in the next section.

It is also known (Ffowcs Williams & Hall 1970) that turbulence located near a sharp edge of an extensive boundary, such as a rigid semi-infinite sheet, provides an even more powerful source of acoustic radiation. The simplest definitive example of this which is amenable to analysis is that discussed by Crighton (1972).

A concentrated line vortex with its axis parallel to the edge of a semi-infinite rigid plate is generated at a distance from the edge which is large compared with the shortest distance of the vortex from the plate. If the vortex strength κ is sufficiently small and the circulation is in the appropriate sense, the vortex moves under the action of an image distribution in the plate along the path illustrated in figure 1 and given by the polar equation

$$R_0 = a\sec(\tfrac{1}{2}\theta_0). \tag{5.11}$$

The motion of the vortex is essentially steady when it is located far from the edge, but sound is emitted as it passes around the edge. We assume that κ is small enough that a typical acoustic wavelength is always large compared with the distance of the vortex from the edge of the plate. The dominant instantaneous frequency is determined by the angular velocity $\dot{\theta}_0$ of the motion, and it follows easily that the *compactness* condition imposes the following restriction on the characteristic Mach number:

$$U/c \ll 1, \tag{5.12}$$

where $U = \kappa/4\pi a$ and a is the distance of closest approach of the vortex to the edge of the plate.

With the x_3 axis parallel to the edge of the plate (directed out of the paper in figure 1), denote the position of the vortex in the 1, 2 plane at time t by $\mathbf{x}_0(t)$, so

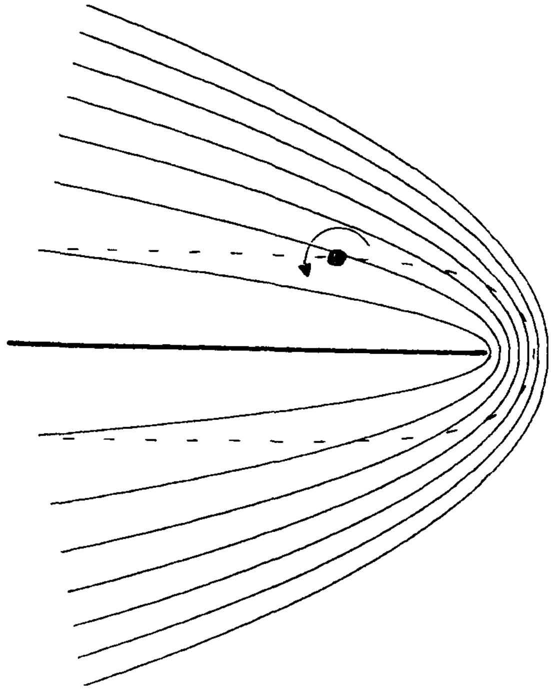

FIGURE 1. A vortex filament negotiates the broken-line path about the edge of a rigid semi-infinite half-plane occupying $x_2 = 0$, $x_1 < 0$. The instantaneous intensity of the radiated sound is proportional to the rate at which the vortex cuts across the hypothetical field of parabolic streamlines which describes irrotational flow about the half-plane.

that the vorticity vector is $\boldsymbol{\omega} = \kappa \mathbf{k} \delta[\mathbf{x}-\mathbf{x}_0(t)]$. The vortex velocity $\mathbf{v} = \dot{\mathbf{x}}_0(t)$, and in the low Mach number approximation the acoustic field is determined by (5.2), where

$$\boldsymbol{\omega} \wedge \mathbf{v} = \kappa \mathbf{k} \wedge \dot{\mathbf{x}}_0(t) \,.\, \delta\{\mathbf{x}-\mathbf{x}_0(t)\}. \tag{5.13}$$

The low-frequency two-dimensional scattering Green's function for this problem is determined in the appendix, and is given by

$$G(\mathbf{x},\mathbf{y};t,\tau) = \frac{\phi^*(\mathbf{x})\,\phi^*(\mathbf{y})}{\pi R}\,\delta\left\{t-\tau-\frac{R}{c}\right\}, \tag{5.14}$$

where the observation point $\mathbf{x}$ in the 1, 2 plane is located many wavelengths from the edge of the half-plane, $R = |\mathbf{x}|$ and

$$\phi^*(\mathbf{x}) = R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta \tag{5.15}$$

is a potential function which describes irrotational incompressible flow about the half-plane.

Applying this to the dipole source term (5.13), the far-field acoustic perturbation pressure is given by

$$\frac{p}{\rho_0} \simeq \frac{\phi^*(\mathbf{x})}{\pi R} \int \frac{\partial}{\partial \mathbf{y}} \,.\, \{\kappa \mathbf{k} \wedge \dot{\mathbf{x}}_0(\tau)\,\delta(\mathbf{y}-\mathbf{x}_0(\tau))\}\,\phi^*(\mathbf{y})\,\delta\left\{t-\tau-\frac{R}{c}\right\} d^2\mathbf{y}\,d\tau$$

$$= -\frac{\kappa\phi^*(\mathbf{x})}{\pi R}[\mathbf{k}\,.\,\dot{\mathbf{x}}_0 \wedge \nabla\phi^*], \tag{5.16}$$

where the quantity in square brackets is evaluated at the retarded position $\mathbf{x}_0(t-R/c)$ of the vortex.

Now $\dot{\mathbf{x}}_0 \wedge \nabla\phi^* = -(\dot{\mathbf{x}}_0 . \nabla\psi)\,\mathbf{k}$, where ψ is the stream function conjugate to ϕ^* for potential flow about the half-plane. Hence

$$\frac{p}{\rho_0} \simeq \frac{\kappa\phi^*(\mathbf{x})}{\pi R}[\dot{\mathbf{x}}_0 . \nabla\psi], \tag{5.17}$$

a result which can also be expressed in the form

$$p \simeq \frac{\rho_0 \kappa \sin\frac{1}{2}\theta}{\pi R^{\frac{1}{2}}}\left[\frac{D\psi}{Dt}\right]. \tag{5.18}$$

It is a simple matter to confirm that this agrees with the prediction based on matched expansions obtained by Crighton (1972). The formula (5.18) is more explicit than Crighton's result, however, and reveals that sound is generated only during the period in which the vortex is cutting across the streamlines of a hypothetical potential flow about the sharp edge. The vortex moves parallel to these streamlines when it is far from the edge, and no sound is generated. In the vicinity of the edge the vortex path departs considerably from that of the potential flow, and the *rate* at which the vortex traverses the streamlines determines the instantaneous intensity of the scattered sound. This radiation mechanism is analogous to the generation of electromagnetic waves which occurs when an electrical conductor cuts across magnetic lines of force.

The $\sin\frac{1}{2}\theta$ angular dependence on the observation position in (5.18) is typical of radiation directivities associated with half-plane scattering problems. Also, as Crighton has pointed out, the total energy radiated by the vortex during its passage about the edge is proportional to the third power of the velocity, which is typically $O(M^{-3})$ greater than that radiated during the characteristic lifetime of a two-dimensional eddy located in free space.

6. Low Mach number convection of turbulence past scattering bodies

We now proceed to examine more general situations in which turbulence is convected in a low Mach number, irrotational mean flow in the neighbourhood of a rigid body. The theory of compact scattering bodies interacting with stationary elements of turbulence is well known and is documented in Ffowcs Williams (1969). At low turbulence Mach numbers the mean-square radiated pressure level varies parametrically with the sixth power of the turbulent velocity fluctuation. The effect of convection past a rigid body is essentially different from the Doppler amplification which occurs when acoustic sources are convected in free space. Indeed, in view of the work of Morfey (1973) and Ffowcs Williams & Howe (1975) on the noise generated during the accelerated motion of nominally silent entropy inhomogeneities, there are serious grounds for believing that the unsteady motion of the turbulent element will generate sound possibly of more significance than has hitherto been supposed, especially in the context of excess jet noise.

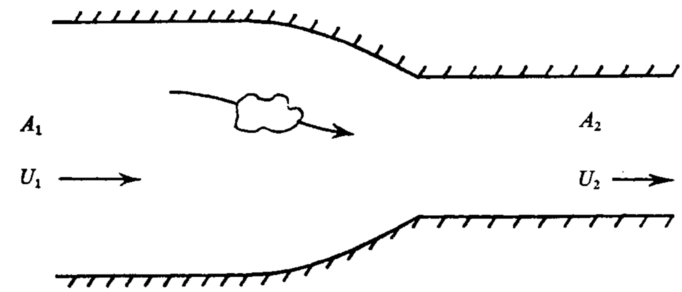

FIGURE 2. A turbulent eddy is convected in a low Mach number, irrotational, steady flow through a contraction in a hard-walled duct of infinite extent.

We shall consider isentropic flows in which the only important source term is the Powell dipole $\operatorname{div}(\boldsymbol{\omega} \wedge \mathbf{v})$ on the right of (4.12). The turbulent element is assumed to be compact and the mean irrotational flow field to have a Mach number M satisfying $M^2 \ll 1$. This implies that the mean flow is effectively incompressible with a constant speed of sound. Under these circumstances (4.12) may be approximated by the convected wave equation

$$\left\{\frac{1}{c^2}\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)^2-\nabla^2\right\}B = \operatorname{div}(\boldsymbol{\omega}\wedge\mathbf{v}), \tag{6.1}$$

where the mean flow velocity satisfies $\operatorname{div}\mathbf{U} = 0$.

Consider first the problem depicted in figure 2. A hard-walled duct of infinite extent contains fluid of density ρ_0 in a state of steady incompressible irrotational flow. The flow is in the $+x_1$ direction and accelerates through a contraction over which the uniform cross-sectional area of the duct reduces from A_1 to A_2 and the mean flow velocity increases from U_1 to U_2, with $A_1 U_1 = A_2 U_2$.

A low Mach number turbulent eddy is convected in the mean flow and it is required to determine the sound which is radiated downstream as the turbulence accelerates through the contraction, a problem which may be regarded as modelling the generation of sound far upstream of a nozzle. In the case of a duct of uniform cross-sectional area, it is known (Ffowcs Williams 1969) that the intensity of the acoustic radiation within the duct is proportional to the sixth power of the characteristic turbulent fluctuation velocity. We shall be interested in field strengths in excess of this, associated with the effect of acceleration through the contraction.

The eddy is assumed to remain compact during this process. The velocity within the eddy has two components:

$$\mathbf{v} = \mathbf{U}+\mathbf{u}; \tag{6.2}$$

$\mathbf{u}$ is the fluctuation velocity induced by the vorticity and effects of images in the walls of the duct, and satisfies $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$.

Equation (6.1) can be solved with the aid of the low frequency Green's function given in equation (A 16) of the appendix, provided that the acoustic wavelengths involved are large compared with the diameter of the duct. This is certainly the case for sufficiently low Mach numbers. Thus, for an observation point $\mathbf{x}$ located many wavelengths downstream of the contraction, the perturbation stagnation enthalpy is given by

$$B = \frac{c}{A_1+A_2}\int \operatorname{div}\{(\boldsymbol{\omega}\wedge\mathbf{v})(\mathbf{y},\tau)\}H\left\{t-\tau-\frac{x_1}{c(1+M_2)}+\frac{A_1}{A_2}\frac{\phi^*(\mathbf{y})}{c(1+M_1)}\right\}d^3\mathbf{y}\,d\tau$$
$$\simeq \frac{-A_1}{A_2(A_1+A_2)(1+M_1)}\int[\boldsymbol{\omega}\wedge\mathbf{v}\,.\,\nabla\phi^*]\,d^3\mathbf{y}, \qquad (6.3)$$

where a quantity in square brackets is evaluated at the retarded time

$$t-x_1/c(1+M_2).$$

Observing that $\mathbf{U}$ is proportional to $\nabla\phi^*$, and using the vector identity $\boldsymbol{\omega}\wedge\mathbf{u} = \mathbf{u}\,.\,\nabla\mathbf{u}-\nabla(\tfrac{1}{2}u^2)$ and the low Mach number approximation $\operatorname{div}\mathbf{u} = 0$, equation (6.3) can be reduced to

$$B \simeq \frac{-A_1}{A_2(A_1+A_2)(1+M_1)}\int\left[\frac{\partial}{\partial y_i}(u_iu_j)-\frac{\partial}{\partial y_j}(\tfrac{1}{2}u^2)\right]\frac{\partial\phi^*}{\partial y_j}\,d^3\mathbf{y}$$
$$= \frac{A_1}{A_2(A_1+A_2)(1+M_1)}\int\left[u_iu_j\frac{\partial^2\phi^*}{\partial y_i\,\partial y_j}\right]d^3y, \qquad (6.4)$$

since $\nabla^2\phi^* = 0$. In the appendix it is shown that $\phi^*(\mathbf{x})\to+x_1$ as $x_1\to+\infty$, so that we may write

$$\frac{\partial^2\phi^*}{\partial y_i\,\partial y_j} = \frac{1}{2U_2}\left[\frac{\partial U_i}{\partial y_j}+\frac{\partial U_j}{\partial y_i}\right]. \qquad (6.5)$$

Also, for $x_1 \gg L$, the scale of the duct contraction, the acoustic pressure and the perturbation stagnation enthalpy are related by

$$B \simeq (1+M_2)\,p/\rho_0, \qquad (6.6)$$

since the perturbation velocity is just equal to $p/\rho_0 c$. Hence acoustic pressure waves radiated downstream have the form

$$p = \frac{1}{2}\left(\frac{A_1}{A_1+A_2}\right)\frac{1}{(1+M_1)(1+M_2)}\frac{1}{A_2U_2}\int\left[\rho_0u_iu_j\left(\frac{\partial U_i}{\partial y_j}+\frac{\partial U_j}{\partial y_i}\right)\right]d^3\mathbf{y}. \qquad (6.7)$$

This expression reveals that the sound may be regarded as generated as a consequence of the work done by the turbulent Reynolds stress $\rho_0u_iu_j$ in the rate-of-strain field $\partial U_i/\partial y_j+\partial U_j/\partial y_i$ of the mean flow. The result (6.7) remains valid even in the *absence* of a mean flow ($U_2 = 0$), since the straining field is actually a geometrical property of the duct in much the same way as the hypothetical streamlines were a property of the half-plane considered in § 5.

For sufficiently high convection speeds the turbulence is effectively *frozen* as it passes through the contraction, and sound is generated as the *steady* Reynolds stress is swept through the variable rate-of-strain field of the mean flow.

Note that the mean-square radiated pressure level predicted by (6.7) is $O(u^4)$, which is $O(M^{-2})$ greater than that generated by turbulence in a uniform duct. Also (6.7) vanishes identically when the turbulence is located sufficiently far from the contraction, where the straining mechanism ceases to be important. A frozen turbulent field generates no sound at such points; for an evolving eddy

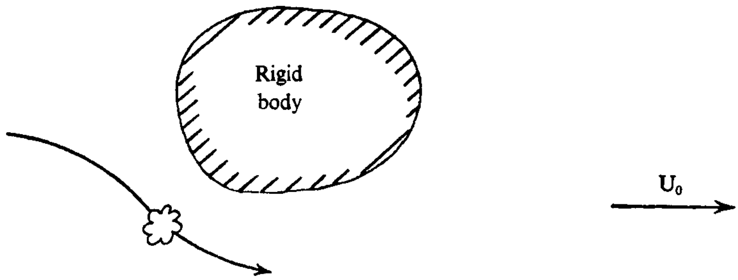

FIGURE 3. A turbulent eddy is convected in a low Mach number, irrotational, steady flow past a fixed solid in free space.

it is necessary to proceed to the next approximation in order to determine the $O(M^2u^4)$ mean-square radiated pressure levels already discussed.

The remaining principal effect of the mean flow in (6.7) arises from the mixed Doppler factor $(1+M_1)(1+M_2)$. Thus there is no significant modification of the amplitude of the acoustic field due to flow since the scattered sound always scales on the eddy fluctuation velocity $\mathbf{u}$ rather than that of the mean flow. To examine this question further we next consider the case of the convection of turbulence past a compact rigid body in free space.

A compact turbulent eddy in free space radiates according to Lighthill's u^8 law, and when it is located in the neighbourhood of a compact body in the absence of a mean flow the radiation is proportional to u^6 (Ffowcs Williams 1969). In figure 3 a turbulent eddy is convected in an irrotational incompressible flow past the rigid body; the flow velocity is equal to $\mathbf{U}_0$ at large distances from the body. In this case (6.1) can be solved by means of the low frequency Green's function given in equation (A 13) of the appendix.

Thus the scattered radiation has the form

$$B \simeq \frac{1}{4\pi|\mathbf{x}|}\int \operatorname{div}\{\boldsymbol{\omega}\wedge\mathbf{v}(\mathbf{y},\tau)\}\,\delta\left\{t-\tau-\frac{|\mathbf{x}-\mathbf{Y}|}{c}+\frac{\mathbf{M}_0.(\mathbf{x}-\mathbf{Y})}{c}\right\}d^3\mathbf{y}\,d\tau$$
$$\simeq -\frac{1}{4\pi c|\mathbf{x}|}\frac{\partial}{\partial t}\int\left[\boldsymbol{\omega}\wedge\mathbf{v}\,.\nabla\left\{\left(\frac{\mathbf{x}}{|\mathbf{x}|}-\mathbf{M}_0\right).\mathbf{Y}\right\}\right]d^3\mathbf{y}, \tag{6.8}$$

where Y_i is the potential of incompressible flow about the body in which the flow at great distances from the body is of unit speed in the i direction, and where, in the notation of the previous problem, $\mathbf{v} = \mathbf{U}+\mathbf{u}$. The quantities in square brackets are evaluated at the retarded time $t-(|\mathbf{x}|-\mathbf{M}_0.\mathbf{x})/c$, the origin of the co-ordinate system being located close to the body.

Introduce the dimensionless velocity

$$U_i^* = \frac{\partial}{\partial y_i}\left\{\left(\frac{\mathbf{x}}{|\mathbf{x}|}-\mathbf{M}_0\right).\mathbf{Y}\right\}. \tag{6.9}$$

This quantity describes an irrotational flow about the body and tends asymptotically to $\mathbf{x}/|\mathbf{x}|-\mathbf{M}_0$ at large distances, where it is actually parallel to the normal to the wave fronts arriving at the observation point $\mathbf{x}$. It is the only term in (6.8) which characterizes the instantaneous directivity of the scattered sound, which is seen to be that of a dipole.

As before, the contribution to (6.8) from the rotational component $\mathbf{u}$ of the eddy velocity $\mathbf{v}$ can be expressed in the form

$$B \simeq \frac{1}{8\pi c|\mathbf{x}|} \frac{\partial}{\partial t} \int \left[u_i u_j \left(\frac{\partial U_i^*}{\partial y_j} + \frac{\partial U_j^*}{\partial y_i} \right) \right] d^3\mathbf{y}. \tag{6.10}$$

Noting that $B \simeq (p/\rho_0)(1 + \mathbf{M}_0 . \mathbf{x}/|\mathbf{x}|)$ and placing the partial time derivative under the integral sign, we find

$$p \simeq \frac{1}{8\pi c|\mathbf{x}|(1 + \mathbf{M}_0 . \mathbf{x}/|\mathbf{x}|)} \int \left[\frac{D}{Dt} \left\{ \rho_0 u_i u_j \left(\frac{\partial U_i^*}{\partial y_j} + \frac{\partial U_j^*}{\partial y_i} \right) \right\} \right] d^3\mathbf{y}. \tag{6.11}$$

This contribution to the sound field is analogous to the result (6.7) obtained in the duct problem. Again the radiation is due to the effective rate of working of the turbulent Reynolds stress in a rate-of-strain field characteristic of the shape of the body. Radiation also occurs when the turbulent field is effectively frozen as it convects past the body. The mean-square acoustic pressure levels vary in proportion to $u^4 M_0^2$, M_0 being the mean flow Mach number.

The contribution to (6.8) which arises from the mean convection velocity component $\mathbf{U}$ of $\mathbf{v}$ can similarly be reduced to the form

$$p \simeq \frac{-\rho_0}{4\pi c|\mathbf{x}|(1 + \mathbf{M}_0 . \mathbf{x}/|\mathbf{x}|)} \int \left[\frac{D}{Dt} \left\{ \boldsymbol{\omega} \wedge \mathbf{U} . \nabla \left(\frac{\mathbf{x} . \mathbf{Y}}{|\mathbf{x}|} \right) \right\} \right] d^3\mathbf{y}. \tag{6.12}$$

When the radiation direction $\mathbf{x}/|\mathbf{x}|$ is parallel to $\mathbf{U}_0$, the vector $\nabla(\mathbf{x} . \mathbf{Y}/|\mathbf{x}|)$ is parallel to the local mean velocity $\mathbf{U}$ and the integrand in (6.12) vanishes identically, and consequently there is *no* radiation from this component of the scattered field in the directions parallel to the mean flow. In other words the dipoles distributed over the surface of the solid are always arranged in such a manner that the resultant dipole has no component in the mean stream direction. This accounts for the absence of this term from the duct problem, since a dipole source can radiate in a duct only if the axis of the dipole has a component parallel to the duct. The mean-square acoustic pressure determined by (6.12) is proportional to $u^2 U_0^2 M_0^2$, which exceeds the contribution from (6.11) provided that $U_0 > u$. Note also that when the turbulent eddy is located far from the body the terms $\mathbf{U}$ and $\nabla(\mathbf{x} . \mathbf{Y}/|\mathbf{x}|)$ in the square brackets in (6.12) are constant, and radiation occurs only if $D\boldsymbol{\omega}/Dt$ is non-zero. Since the eddy is essentially incompressible, $D\omega_i/Dt = \partial(\omega_j v_i)/\partial x_j$, a divergence which integrates to zero in the approximation of (6.12). Thus at these points it is necessary to take the next approximation to the Green's function, in which case we recover the Lighthill U^8 radiation intensity law. Similar remarks apply to the result (6.11).

The above analysis can be applied qualitatively to the problem of sound generation by a turbulent wake if the mean flow Mach number is sufficiently small. In this case the rotational velocity $\mathbf{u}$ is comparable with the mean flow velocity, although the translational velocity of a typical vortical core is small until vortex shedding occurs. When this happens the core accelerates to a velocity comparable with that of the mean flow and a sound pulse is radiated. The periodic nature of these events is responsible for the characteristic Aeolian tones at the vortex shedding frequency (see, for example, Blokhintsev 1946, p. 112). Our analysis indicates that the dominant radiation is in directions at right angles to the mean flow. This is in accordance with the observed dipole nature of the Aeolian tone. The dipole axis is perpendicular to the mean flow and corresponds to a fluctuating lift force, the mean-square radiated pressure varying as $U_0^4 M_0^2$.

7. Generation of sound by entropy inhomogeneities

The presence of entropy inhomogeneities in a jet flow, produced, for example, by non-uniform combustion processes, is responsible for the generation of sound when the inhomogeneities accelerate in the pressure gradient of the mean flow (Morfey 1973). Candel (1972) and Marble (1973) examined this mechanism in quasi-one-dimensional flows by first decomposing the inhomogeneities into a spectrum of harmonic entropy waves. Two methods of analysis were employed. The first considered the interaction of each entropy wave with *compact* elements of the mean flow (e.g. a contraction in a duct), a procedure which is expected to be valid when the lengths and transit times are small compared with the wavelength and wave period. The second approach was based on the assumption that the variation in the mean flow parameters could be regarded as one-dimensional. It is not possible, however, to treat by these methods problems involving the convection of sharp-fronted entropy *spots*, in which the characteristic scale of the entropy variation is much *smaller* than that of the mean flow.

Ffowcs Williams & Howe (1975) considered the problem of sound generation when an entropy *slug* completely filling a compact section of a duct is convected in a low Mach number flow through a contraction in the duct. The analysis involved an application of the generalized Kirchhoff theorem developed in Ffowcs Williams & Hawkings (1969). Unfortunately their method cannot easily be extended to deal rigorously with the more general problem of entropy spots.

It is actually possible to handle these problems with relative ease using the general equations (4.12) and (4.14). Confining attention to low Mach number flow situations, and neglecting the contribution to the radiated sound due to the presence of vorticity in the flow, the appropriate approximation to the acoustic-analogy equation assumes the form

$$\left\{\frac{1}{c^2}\left(\frac{\partial}{\partial t}+\mathbf{U}.\frac{\partial}{\partial \mathbf{x}}\right)^2-\nabla^2\right\}B=-\operatorname{div}\left(T\operatorname{grad}S\right), \tag{7.1}$$

where heat conduction within the fluid has also been neglected.

Specification of an entropy spot

In the absence of thermal conduction and viscous dissipation we shall consider a low Mach number mean flow which is isentropic, irrotational and of density ρ_0. We consider an entropy spot in the form of a region of fluid bounded by a closed surface $f(\mathbf{x}, t) = 0$ within which the specific entropy is constant and different from that in the mean flow (figure 4). The pressure must be continuous across the bounding surface, which is therefore characterized by a jump in the temperature and density of the fluid. Using the thermodynamic relation

$$TdS=\frac{1}{\gamma-1}\frac{dp}{\rho}+\frac{\gamma p}{\gamma-1}d\left(\frac{1}{\rho}\right) \tag{7.2}$$

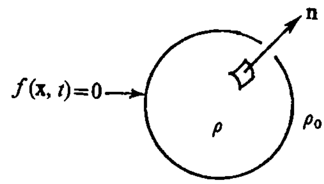

FIGURE 4. The entropy inhomogeneity is bounded by the surface $f(\mathbf{x}, t) = 0$, across which the pressure is continuous, but the temperature and density of the fluid are discontinuous. The specific entropy is uniform within the entropy spot, and the ambient flow is isentropic.

for an ideal gas, it follows that the discontinuity across the surface is specified by

$$T \operatorname{grad} S = \frac{\gamma p}{\gamma - 1} \operatorname{grad}\left(\frac{1}{\rho}\right). \tag{7.3}$$

Thus, if ρ is the fluid density within the spot and $\Delta\rho = \rho - \rho_0$, then

$$\begin{aligned} T \operatorname{grad} S &= \frac{\gamma}{\gamma-1}\frac{p}{\rho_0}\left(\frac{\Delta\rho}{\rho}\right)|\nabla f|\,\delta(f)\,\mathbf{n} \\ &= \left(\frac{\Delta\rho}{\rho}\right)|\nabla f|\,\delta(f)\,\mathbf{n}\int\frac{dp}{\rho_0}, \end{aligned} \tag{7.4}$$

since

$$\int\frac{dp}{\rho_0} = \frac{\gamma}{\gamma-1}\frac{p}{\rho_0}$$

in an ideal gas, where $\mathbf{n}$ is the unit normal to the surface $f(\mathbf{x}, t) = 0$ illustrated in figure 4.

Hence (7.1) becomes

$$\left\{\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \mathbf{U}.\frac{\partial}{\partial \mathbf{x}}\right)^2 - \nabla^2\right\}B = -\operatorname{div}\left[\left(\frac{\Delta\rho}{\rho}\right)|\nabla f|\,\delta(f)\,\mathbf{n}\int\frac{dp}{\rho_0}\right]. \tag{7.5}$$

Convection in ducts and past rigid bodies

We now apply (7.5) to problems of the type already considered in connexion with convected turbulence inhomogeneities in § 6.

Consider first the model in figure 2, in which an entropy spot is convected by the mean flow through a contraction in a duct. The source term on the right of (7.5) is time dependent only in the vicinity of the contraction, which is therefore the source of the acoustic radiation. The typical time scale of the unsteady motion is of order L/u and at low mean flow Mach numbers the corresponding acoustic wavelengths are large, so that (7.5) can be solved using the low frequency Green's function of equation (A 16).

Thus at an observation point $\mathbf{x}$ located many wavelengths downstream of the contraction we have

$$\begin{aligned} B &= \frac{-c}{(A_1+A_2)}\int\frac{\partial}{\partial y_j}\left\{\left(\frac{\Delta\rho}{\rho}\right)|\nabla f|\,\delta(f)\,n_j\int\frac{dp}{\rho_0}\right\}H\left\{t-\tau-\frac{x_1}{c(1+M_2)}+\frac{A_1}{A_2}\frac{\phi^*(\mathbf{y})}{c(1+M_1)}\right\}d^3\mathbf{y}\,d\tau \\ &= \frac{A_1}{A_2(A_1+A_2)(1+M_1)}\oint\left[\left(\frac{\Delta\rho}{\rho}\right)\int\frac{dp}{\rho_0}\mathbf{n}.\nabla\phi^*\right]d^2\mathbf{y}, \end{aligned} \tag{7.6}$$

where the second integral is taken over the bounding surface of the entropy spot and the quantity in square brackets is evaluated at the retarded time $t - x_1/c(1+M_2)$. Transforming this integral by means of the divergence theorem, and recalling that $\nabla^2\phi^* = 0$ and that $\Delta\rho/\rho$ may be assumed constant in the first approximation, we have

$$B \simeq \frac{A_1}{A_2(A_1+A_2)(1+M_1)}\left(\frac{\Delta\rho}{\rho}\right)\left[\int_\sigma \frac{\nabla\phi^*.\nabla p}{\rho_0}d^3\mathbf{y}\right], \tag{7.7}$$

the integration being over the volume σ of the entropy spot.

If the diameter of the entropy spot is small compared with the scale of variation of the mean pressure gradient and $\Delta\rho/\rho$ is small it follows from (7.7) in the usual manner that the acoustic pressure perturbation far downstream of the contraction is given by

$$p \simeq \frac{\sigma A_1}{(A_1+A_2)(1+M_1)(1+M_2)}\left(\frac{\Delta\rho}{\rho}\right)\frac{1}{A_2 U_2}\left[\frac{Dp}{Dt}\right], \tag{7.8}$$

the material derivative being evaluated at the retarded position of the entropy spot.

This result displays the same dependence on the area ratio and mean flow Doppler factors as the corresponding formula (6.7) for a vortical inhomogeneity. However the mean-square radiated pressure level now scales on U^4, where U is a characteristic *mean* flow speed. The presence of the material derivative confirms the prediction that radiation occurs only where the mean flow pressure field varies along the trajectory of the inhomogeneity. We also note that our general formula (7.6) is in agreement with the result obtained by Ffowcs Williams & Howe (1975) by a different method for the particular case of an entropy *slug*.

The above calculation can be repeated for the case of low Mach number convection in an irrotational flow past a solid obstacle in free space (figure 3). The procedure follows closely the method outlined in the previous section, and it is readily deduced that the scattered acoustic enthalpy perturbation is given by

$$\begin{aligned} B &\simeq -\frac{1}{4\pi|\mathbf{x}|}\int\frac{\partial}{\partial y_j}\left\{\left(\frac{\Delta\rho}{\rho}\right)|\nabla f|\delta(f)n_j\int\frac{dp}{\rho_0}\right\}\delta\left\{t-\tau-\frac{|\mathbf{x}-\mathbf{Y}|}{c}+\frac{\mathbf{M}_0.(\mathbf{x}-\mathbf{Y})}{c}\right\}d^3\mathbf{y}\,d\tau \\ &\simeq \frac{1}{4\pi c|\mathbf{x}|}\left(\frac{\Delta\rho}{\rho}\right)\frac{\partial}{\partial t}\oint\left(\frac{x_i}{|\mathbf{x}|}-M_{0i}\right)\left[n_j\frac{\partial Y_i}{\partial y_j}\int\frac{dp}{\rho_0}\right]d^2\mathbf{y}. \end{aligned} \tag{7.9}$$

The second integral in (7.9) is taken over the bounding surface of the entropy spot, the quantity in square brackets being evaluated at the retarded time $t - |\mathbf{x}|/c + \mathbf{M}_0.\mathbf{x}/c$.

Using the divergence theorem, and introducing the dimensionless velocity U_i^* defined in (6.9), the scattered acoustic pressure field can be expressed in the form

$$p \simeq \frac{\sigma}{4\pi c|\mathbf{x}|(1+\mathbf{M}_0.\mathbf{x}/|\mathbf{x}|)}\left(\frac{\Delta\rho}{\rho}\right)\left[\frac{D}{Dt}(\mathbf{U}^*.\nabla p)\right]. \tag{7.10}$$

This formula shows that when $\Delta\rho/\rho$ is small the mean-square acoustic pressure scales on $(\Delta\rho/\rho)^2 M_0^2 U_0^4$, or equivalently $(\Delta T/T)^2 M_0^2 U_0^4$, where ΔT is the temperature difference between the entropy spot and the ambient flow, and indicates that this noise-producing mechanism is possibly more significant than pure jet noise at low subsonic flow speeds.

When the formulae (6.11), (6.12) and (7.10) are applied to the case of a rigid body translating at uniform velocity in a fluid which is at rest at infinity, so that the interaction involved is with a nominally stationary flow inhomogeneity, it is appropriate to express the radiated field in terms of the position of the observation point relative to the body at the time of *emission* of the sound. The results are clearly recognizable as *dipole* radiation fields augmented by *three* powers of the Doppler factor $(1-M_r)^{-1}$, M_r being the component of the Mach number of the velocity of the body in the direction of emission of the sound. This contrasts with the *two* Doppler factors associated with a translating *point* dipole source, and has also been observed in a different context by Howe (1975), Crighton (private communication) and Ffowcs Williams & Lovely (1975).

8. Sound generation by fluctuating heat sources

In this section we examine the validity of the acoustic-analogy equation (4.14) in situations where it is not permissible to neglect the conduction of heat in the fluid, although we shall still assume that viscous effects are unimportant in the source region. The relaxation of the condition of no thermal conductivity implies that the entropy of a fluid particle is not necessarily constant in time. As a fluid element moves within the flow it will absorb heat energy at a rate equal to $\rho TDS/Dt$ per unit volume. In the absence of viscous dissipation this can also be expressed in terms of the temperature gradient within the fluid by means of

$$\rho TDS/Dt = \operatorname{div}(K \operatorname{grad} T), \tag{8.1}$$

where K is the *thermal conductivity* of the fluid (Landau & Lifshitz 1959, p. 185). Consequently all of the entropy source terms on the right of (4.14) are now of significance.

We proceed to consider the simple model problem of the generation of sound by periodic temperature fluctuations of a body immersed in a fluid at rest. The problem can also be solved by an alternative procedure, and the calculation will thereby provide a tentative check on the validity of (4.14).

The surface temperature of a rigid body immersed in the fluid is caused to vary periodically at a radian frequency ω. If there is no mean flow (8.1) implies that the oscillations in the temperature are communicated to the fluid in the form of *thermal waves* which are rapidly damped within a boundary-layer region whose width is of order $(\chi/\omega)^{\frac{1}{2}}$, where $\chi = K/\rho_0 c_p$ is the *thermometric conductivity* of the fluid. This heating is accompanied by a periodic expansion and contraction of the fluid within the boundary layer, as a result of which there is a pulsating mass flux through a mathematical control surface which just encloses the body and the boundary layer. These pulsations give rise to a sound wave in the distant field, the body behaving as a *monopole* source.

When the acoustic wavelength greatly exceeds the boundary-layer width, i.e. when $\omega \ll c^2/\chi$, the characteristics of the sound can be determined in two stages. In the neighbourhood of the body the fluid may be assumed to be incompressible, the density variations arising solely from the changes in volume due to periodic heating essentially at constant pressure. The periodic mass flux through the outer edge of the boundary layer can then be determined by means of the equation of continuity and matched onto a radiating sound wave (Landau & Lifshitz 1959, p. 287). Our object here is to obtain the acoustic field directly from (4.14).

Now the equations of continuity and (8.1) together imply that the velocity fluctuation at the outer limit of the boundary layer is of order $(\omega\chi)^{\frac{1}{2}}(\Delta T/T)$, where ΔT is the amplitude of the temperature fluctuations in the body and T is the mean temperature. In order to ensure that acoustic propagation is correctly described by a linearized wave operator we shall therefore assume that variations in the thermodynamic quantities are of first order. Thus, neglecting the vorticity terms on the right of (4.14) and linearizing with respect to the thermodynamic variables, the acoustic equation becomes

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right)+\frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot\frac{\partial}{\partial\mathbf{x}}-\nabla^2\right\}[B-TS'] = \frac{\partial}{\partial t}\left(\frac{1}{c_p}\frac{DS'}{Dt}\right), \tag{8.2}$$

where S' denotes the fluctuating component of the specific entropy. In the distant field S' vanishes identically, and in the absence of a mean flow $B \sim p/\rho_0$. Also, since in the same approximation Crocco's form of the momentum equation (4.4) is

$$\partial\mathbf{v}/\partial t+\operatorname{grad}(B-TS') = 0, \tag{8.3}$$

it follows that the boundary condition to be satisfied by $B-TS'$ on the fixed surface of the body is the usual one of vanishing normal derivative.

Since there is no mean flow (8.2) may be approximated further by

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\frac{p}{\rho_0} = \frac{\partial}{\partial t}\left(\frac{1}{c_p}\frac{\partial S'}{\partial t}\right), \tag{8.4}$$

where $B-TS'$ has been replaced by its far-field representation in terms of the acoustic pressure perturbation p.

Consider the case of a plane rigid wall whose temperature varies periodically with frequency $\omega \ll c^2/\chi$. Neglecting the small fluctuations in pressure near the wall (cf. Landau & Lifshitz 1959, p. 287), (8.1) can be used to show that

$$S' = c_p(\Delta T/T)\exp\{-i\omega t-(1-i)\,x_1(\omega/2\chi)^{\frac{1}{2}}\}, \tag{8.5}$$

where the x_1 direction is normal to the wall. The acoustic waves are one-dimensional, and the appropriate Green's function for (8.4) which satisfies the condition of vanishing normal derivative on $x_1 = 0$ is

$$G(x_1,y_1;t,\tau) = \tfrac{1}{2}c\{H(t-\tau-|x_1-y_1|/c)+H(t-\tau-|x_1+y_1|/c)\}. \tag{8.6}$$

Hence the acoustic pressure field is given by

$$\begin{aligned}\frac{p}{\rho_0} &\simeq \int\frac{\partial}{\partial\tau}\left(\frac{1}{c_p}\frac{\partial S'}{\partial\tau}(y_1,\tau)\right)G(x_1,y_1;t,\tau)\,dy_1\,d\tau \\ &= (1-i)\left(\frac{\Delta T}{T}\right)c\left(\frac{\omega\chi}{2}\right)^{\frac{1}{2}}\exp\left\{-i\omega\left(t-\frac{x_1}{c}\right)\right\},\end{aligned} \tag{8.7}$$

provided that $\omega \ll c^2/\chi$. This result agrees *precisely* with that obtained by Landau & Lifshitz (1959, p. 287), who used the method of matched expansions. This provides provisional confirmation of the acoustic-analogy equation (4.14).

The problem of the radiation from a rigid sphere whose temperature varies periodically can also be solved with relative ease. The entropy fluctuation close to the sphere is given by

$$S' = c_p \left(\frac{\Delta T}{T}\right) \frac{R}{|\mathbf{x}|} \exp\{-i\omega t - (1-i)(|\mathbf{x}| - R)(\omega/2\chi)^{\frac{1}{2}}\}, \tag{8.8}$$

where R is the radius of the sphere and the origin of co-ordinates is located at the centre of the sphere. The low frequency Green's function for this problem, which is appropriate if $\omega R/c$ is small, is given in equation (A 13) of the appendix, from which we deduce that the radiated pressure field is given by

$$\frac{p}{\rho_0} = -\frac{\omega^2}{4\pi}\left(\frac{\Delta T}{T}\right)\frac{R}{|\mathbf{x}|}\int_{|\mathbf{y}|>R} \frac{\exp\{-(1-i)(|\mathbf{y}|-R)(\omega/2\chi)^{\frac{1}{2}}\}}{|\mathbf{y}|} \times \exp\{-i\omega(t - |\mathbf{x}|/c + \mathbf{x}\,.\,\mathbf{Y}/c|\mathbf{x}|)\}\, d^3\mathbf{y}, \tag{8.9}$$

where $\mathbf{Y} = \mathbf{y}\{1 + R^3/2|\mathbf{y}|^3\}$.

Hence we have

$$\frac{p}{\rho_0} \simeq -\left(\frac{\Delta T}{T}\right)\left(\frac{R}{|\mathbf{x}|}\right)\frac{\omega R(2\omega\chi)^{\frac{1}{2}}}{(1-i)}\left\{1 + \frac{1}{(1-i)R(\omega/2\chi)^{\frac{1}{2}}}\right\}\exp[-i\omega(t - |\mathbf{x}|/c)]. \tag{8.10}$$

This result illustrates the manner in which the dimensions of a finite body enter the problem. The second term in the curly brackets can be neglected provided that the width of the thermal boundary layer is small compared with the radius R of the sphere.

The method described in this section can also be applied to more sophisticated problems such as the theory of the Rijke tube, in which entropy fluctuations are produced by the acoustically coupled periodic transfer of heat from a hot gauze located within the stream of air in the tube.

9. Scattering of a plane wave by a cylindrical vortex filament

Lighthill (1953) and Kraichnan (1953) examined the scattering of sound by turbulence in the *Born approximation*, and the subsequent extensive development of the theory is reported in Chernov (1960) and Tatarski (1961). The multiple scattering and absorption of long waves has been discussed by Crow (1967), and Howe (1973) has considered the multiple scattering of sound by turbulence in terms of a kinetic equation. Paradoxically, perhaps, the problem of the scattering of a plane sound wave by a cylindrical vortex filament has not been resolved satisfactorily.

The difficulty arises because of the long-range scattering effect associated with the (radius)$^{-1}$ decay of the mean vortex-induced flow. This is manifested by the divergence of the integrals of the Born approximation in certain scattering directions. Müller & Matschat (1959) avoided this difficulty by artificially introducing a finite cut-off radius beyond which the mean flow of the vortex could be ignored. In a recent study by O'Shea (1971) the Born approximation was applied directly with no cut-off and the scattered field was observed to be singular in the forward and backward scattering directions.

In order to clarify the situation from the point of view developed in this paper, we shall consider the case of scattering by a circular cylindrical vortex of radius a and vorticity Ω per unit area. In real fluids the formation of a concentrated vortex core is inhibited by the diffusive action of viscosity. Therefore the specific entropy is generally higher in the core than in the exterior potential flow, and the model we shall adopt will be one in which the specific entropy within the core is uniform but different from that in the isentropic ambient flow.

If heat conduction and viscous effects are neglected during scattering, the stagnation enthalpy is determined by (4.12):

$$\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right)B+\frac{1}{c^2}\frac{D\mathbf{v}}{Dt}.\nabla B-\nabla^2 B = \operatorname{div}\{\boldsymbol{\omega}\wedge\mathbf{v}-T\nabla S\}+\frac{1}{c^2}\frac{D\mathbf{v}}{Dt}.T\nabla S, \quad (9.1)$$

where only the leading vortical term has been retained on the right-hand side. This equation shows that there exist two distinct mechanisms of secondary-wave formation. First, cross-product terms on the left-hand side involving the incident wave B_I, say, and the steady, undisturbed stagnation enthalpy distribution B_0 give rise to a scattered field centred solely on the vortex core. Similarly there is a vortex-centred scattered field arising from the discontinuity in the speed of sound across the bounding surface of the vortex, and from the perturbation of the inhomogeneous terms on the right-hand side of (9.1) produced by the inicident wave. Second, the wave operator involves the mean flow and causes the sound to be *refracted*. However, if the wavelength of the incident sound greatly exceeds the radius a of the vortex, and if the maximum Mach number of the mean flow is small, $U/c \equiv a\Omega/2c \ll 1$, refraction is unimportant, and the effect on propagation of the second term on the left of (9.1) may be neglected and the material derivatives replaced by $\partial/\partial t$.

The incident wave satisfies

$$(c_0^{-2}\,\partial^2/\partial t^2-\nabla^2)\,B_I = 0, \quad (9.2)$$

where c_0 is the speed of sound in the exterior potential-flow region. It follows that the scattered acoustic field B' is determined in the first approximation by the equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)B' = \operatorname{div}\{\boldsymbol{\omega}\wedge\mathbf{v}_I-T_I\nabla S\}-\left(\frac{1}{c^2}-\frac{1}{c_0^2}\right)\frac{\partial^2 B_I}{\partial t^2}, \quad (9.3)$$

where $\mathbf{v}_I$ and T_I are respectively the perturbation velocity and temperature associated with the incident wave.

Let the undisturbed axis of the vortex core lie along the x_3 axis, with the vorticity vector parallel to the corresponding unit vector $\mathbf{k}$, and assume that the incident pressure wave is specified by

$$p = p_I\exp\{i\kappa(x_1-c_0t)\}. \quad (9.4)$$

This wave propagates parallel to the x_1 axis. The source term $T_I \nabla S$ is evaluated from the relations $\nabla S = -c_p \nabla \rho/\rho$ and $T_I = p/\rho c_p$, so that the scattered radiation is given by the solution of

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)B' = \frac{p_I}{\rho_0}\operatorname{div}\left\{\left[\frac{\Omega}{c_0}H(-f)\mathbf{k}\wedge\mathbf{i}-\left(\frac{\Delta\rho}{\rho_0}\right)\nabla H(f)\right]\exp\{i\kappa(x_1-c_0t)\}\right\} + \kappa^2\frac{p_I}{\rho_0}\left(\frac{\Delta\rho}{\rho_0}\right)H(-f)\exp\{i\kappa(x_1-c_0t)\}. \quad (9.5)$$

In this equation ρ_0 is the density in the exterior flow and $\rho_0+\Delta\rho$ that within the core; $f(\mathbf{x}) = 0$ is the equation of the *unperturbed* bounding surface of the vortex, with $f > 0$ in the exterior fluid and $f < 0$ within the core, and the discontinuous function

$$\frac{1}{c^2}-\frac{1}{c_0^2}=\frac{1}{c_0^2}\frac{\Delta\rho}{\rho_0}H(-f).$$

In the distant field the scattered waves are given by

$$p' = \frac{p_I}{4\pi}\int\left\{\frac{\partial}{\partial \mathbf{y}}\cdot\left[\left(\frac{\Omega}{c_0}H(-f)\mathbf{j}-\left(\frac{\Delta\rho}{\rho_0}\right)\nabla H(f)\right)\exp\{i\kappa(y_1-c_0\tau)\}\right] + \kappa^2\left(\frac{\Delta\rho}{\rho_0}\right)H(-f)\exp\{i\kappa(y_1-c_0\tau)\}\right\}\delta[t-\tau-|\mathbf{x}-\mathbf{y}|/c_0]\frac{d^3\mathbf{y}\,d\tau}{|\mathbf{x}-\mathbf{y}|}$$

$$\simeq \frac{ip_I(\kappa a)^2}{4}\left(\frac{2\pi}{\kappa R}\right)^{\frac{1}{2}}\left\{\frac{\Omega}{c_0\kappa}\sin\theta - i\frac{\Delta\rho}{\rho_0}\cos\theta\right\}\exp\left(-i\left[c_0\kappa(t-R/c_0)-\tfrac{1}{4}\pi\right]\right), \quad (9.6)$$

where $R = (x_1^2+x_2^2)^{\frac{1}{2}}$ and the y_3 integration has been performed by the method of stationary phase.

Since the problem is two-dimensional the scattered pressure waves decay as the inverse of the square root of the distance R from the axis of the vortex. In the 1, 2 plane, θ denotes the angle between the $+x_1$ axis and the observation direction. The first term in the curly brackets in (9.6) is a vortically induced *dipole* scattered field with axis perpendicular to the direction of propagation of the incident wave. The second term arises because of the variation of the specific entropy of the fluid in the vortex core, i.e. as a result of dissipation processes. It has the characteristic *dipole* structure already encountered in § 7, and is a consequence of the fact that the acceleration experienced by the core in the pressure gradient of the incident wave is different from what it would have been had the density been uniform. The net acoustic effect is therefore equivalent to the application of a fluctuating body force to the ambient fluid at the vortex axis (cf. Rayleigh 1945, §§ 296, 335).

The long-wave approximation used in deriving (9.3) is analogous to the Rayleigh approximation used in the theory of the scattering by compact bodies in a flow (see, for example, Howe 1975). At shorter incident wavelengths, however, it is still appropriate to regard the source of the scattered waves as residing in the vortex core (provided that the incident wavelength exceeds or is of the same order as the core diameter), but ray theory should be applied to describe propagation in the ambient mean flow. This view of the scattering mechanism finds some support in the beautiful interferograms of Naumann & Hermanns (1973), which show the interaction of a shock wave with a vortex field.

10. An aerodynamic theory of the flute

In this final section we sketch the details of an approximate analysis of the mode of operation of wind instruments characterized by the *flute* and *recorder*. The mechanism by which part of the kinetic energy of a stream of air which is blown across the mouth (or 'embouchure') of the flute is communicated to the resonant oscillations of the air within the instrument is generally thought to be associated with the generation and shedding of vortices. When the thickness of the incident jet of air is small compared with the diameter of the mouth, as in the *flue organ pipe*, vortices are formed because of the *instability* of the jet to small disturbances (Rayleigh 1880). In that case the action of the lip of the mouth on which the jet impinges is similar to that which occurs for the classical *edge tone* (Brown 1937; Curle 1953; Powell 1961*b*; Backus 1970), although the frequency of generation of vortical instabilities is set by the sounding frequency of the pipe rather than by the effective length of the stream of air.

Simple order-of-magnitude estimates based on Rayleigh's (1880) results indicate, however, that in the case of instruments such as the flute and recorder the acoustic frequencies involved are too large for the corresponding characteristic disturbances of the air stream to be unstable, so that the growth of vorticity by the usual edge-tone mechanism tends to be inhibited. This view is consistent with the detailed stroboscopic observations of Coltman (1968), in which the vortices are actually formed at the lip of the mouth. The experiments of Richardson (1931) reveal that a distinct source of vortices is associated with the cross-flow through the mouth produced by the standing acoustic wave within the instrument. Periodic flow separation occurs at the lip, and the resulting free vortex sheets roll up to form periodically shed vortex elements.

In the present discussion we shall neglect completely the possibility of jet instability in the formation of the vortices, and adopt a model based on Richardson's mechanism of vortex formation. We shall also neglect dissipation due to viscous and heat-conduction effects at the walls of the instrument, although experiments indicate that in the case of the flute this is far from insignificant (Coltman 1968). The idealized mathematical model is depicted in figure 5, which shows a cylindrical rigid tube of length l which has an open end B and a mouth A. The diameter of the tube and of the mouth are assumed to be small compared with the relevant acoustic wavelengths. The air stream which excites sound waves within the tube is in the direction AB at a constant speed U. (The flute is actually blown in the crosswise direction, but this has no appreciable influence on the sounding mechanism discussed below.)

The mathematical details of the analysis can be separated into three stages. We shall be dealing with the excitation of sound waves in a flow of negligible Mach number by vorticity located in the vicinity of the mouth A. Thus acoustic waves satisfy the following approximation to (4.12):

$$(c^{-2}\,\partial^2/\partial t^2 - \nabla^2)\,B = \operatorname{div}\,(\boldsymbol{\omega} \wedge \mathbf{v}). \tag{10.1}$$

The vortical source term is determined both by the incident air stream of speed U and also by the cross-flow at the mouth, which in turn depends on the

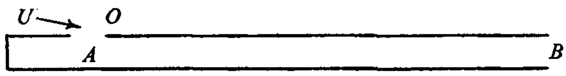

FIGURE 5. Mathematical model of the flute consisting of a hard-walled cylindrical tube open at the end B and the mouth A. The incident air stream of speed U is directed against the lip O.

properties of the acoustic field within the tube. Thus $\mathbf{v}$ and $\boldsymbol{\omega}$ are unknown initially and must be determined during the course of the analysis. This calculation is described in stage II. In order to determine the acoustic field in free space and within the tube, appropriate forms for the Green's function $G(\mathbf{x}, \mathbf{y}; t, \tau)$ for sources located at A must be obtained. This is done in stage I. In the final stage (III) the results of stages I and II are combined to determine the effective source strength on the right of (10.1) and the acoustic radiation in free space.

Stage I: the Green's functions

The acoustic wavelengths involved are always large compared with the diameters of the mouth A and the opening B, so that only the low frequency forms of the Green's functions are required. We first obtain the Green's function for radiation into free space.

By the reciprocal theorem discussed in the appendix we may consider a harmonic source $\delta(\mathbf{x}-\mathbf{y})\, e^{-i\omega t}$ located at the distant point $\mathbf{x}$ in free space, the origin of the co-ordinates being taken in the vicinity of the mouth A. The time-harmonic Green's function $\mathscr{G}(\mathbf{x}, \mathbf{y}; t, \omega)$ is the solution of

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y_i^2}\right)\mathscr{G} = \delta(\mathbf{x}-\mathbf{y})\, e^{-i\omega t} \tag{10.2}$$

subject to the condition of vanishing normal derivative on the solid boundaries and the radiation condition at infinity. The function $\mathscr{G}$ may therefore be regarded as the *potential* of a velocity field, and the reciprocal theorem asserts that the value of $\mathscr{G}$ as a function of $\mathbf{y}$ determined by (10.2) is identical with that which would arise at $\mathbf{x}$ if the source had been located at $\mathbf{y}$.

Let $\mathscr{G}_i$ denote the spherical wave

$$\mathscr{G}_i = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\exp\{-i\omega(t-|\mathbf{x}-\mathbf{y}|/c)\}, \tag{10.3}$$

which is generated at $\mathbf{x}$ and is incident on the flute. For $|\mathbf{x}| \gg |\mathbf{y}|$, and for points $\mathbf{y}$ located in the *near field* of A $(\omega|\mathbf{y}|/c \ll 1)$, (10.3) becomes approximately

$$\mathscr{G}_i \simeq Z_A\{1-i\kappa\mathbf{x}\,.\,\mathbf{y}/|\mathbf{x}|\}, \tag{10.4}$$

where
$$Z_A = \frac{1}{4\pi|\mathbf{x}|}\exp\left\{-i\omega\left(t-\frac{|\mathbf{x}|}{c}\right)\right\}, \quad \kappa = \frac{\omega}{c}. \tag{10.5}$$

The dominant acoustic wavelengths are of the order of the length l of the tube, i.e. $\omega l/c = O(1)$, so that, if $\mathbf{l}$ denotes the vector distance from A to B, then for points $\mathbf{y}$ located in the near field of the orifice at B, the appropriate form for the incident spherical wave is given by

$$\left.\begin{aligned} \mathscr{G}_i &= Z_B\left\{1 - i\kappa\frac{\mathbf{x}.(\mathbf{y}-\mathbf{l})}{|\mathbf{x}|}\right\},\\ Z_B &= \frac{1}{4\pi|\mathbf{x}|}\exp\left\{-i\omega\left(t-\frac{|\mathbf{x}|}{c}+\frac{\mathbf{x}.\mathbf{l}}{c|\mathbf{x}|}\right)\right\}.\end{aligned}\right\} \tag{10.6}$$

Secondary waves are scattered at the tube. In general the scattered field is very weak (Rayleigh scattering) unless the frequency of the incident wave is close to one of the resonant frequencies of the tube, in which case the dominant sources of the scattered radiation are the openings at A and B. We are principally interested in this case, and shall assume that the main component of the scattered sound consists of two spherical waves emanating respectively from A and B. In the vicinity of these points the field must be determined essentially by the properties of an incompressible reciprocating potential flow.

Thus, at points in the exterior fluid located well within an acoustic wavelength of the mouth A but many mouth diameters from A, we can write

$$\mathscr{G} \simeq Z_A\left\{1 - i\kappa\frac{\mathbf{x}.\mathbf{y}}{|\mathbf{x}|}\right\} + \frac{\phi_A(t-r_A/c)}{r_A}, \tag{10.7}$$

where the second term on the right is the scattered spherical wave and $r_A = |\mathbf{y}|$. Differentiation with respect to time is equivalent to multiplication by $-i\omega$, so that (10.7) can also be expressed in the approximate near-field form

$$\mathscr{G} \simeq Z_A\{1 - i\kappa\mathbf{x}.\mathbf{y}/|\mathbf{x}|\} + i\kappa\phi_A + \phi_A/r_A, \tag{10.8}$$

where $\phi_A \equiv \phi_A(t)$.

At A a different form for $\mathscr{G}$ must be employed to take account of the precise details of the essentially incompressible flow at the mouth. Thus we write

$$\mathscr{G} \simeq Z_A\left\{1 - i\frac{\kappa\mathbf{x}}{|\mathbf{x}|}.[\mathbf{y}+\boldsymbol{\phi}_1^*(\mathbf{y})]\right\} + \alpha\phi_2^*(\mathbf{y}) + \phi_0^A(t), \tag{10.9}$$

where $\mathbf{y}+\boldsymbol{\phi}_1^*(\mathbf{y})$ is a potential function whose normal derivative vanishes on the walls of the tube and which satisfies

$$\mathbf{y}+\boldsymbol{\phi}_1^*(\mathbf{y}) \to \left\{\begin{array}{ll} \mathbf{y}+O(r_A^{-2}) & \text{in free space,}\\ \boldsymbol{\gamma}^A\{y_1+\mathscr{A}/K_1\} & \text{at points within the tube many}\\ & \text{mouth diameters from } A, \end{array}\right\} \tag{10.10}$$

where the y_1 axis is taken in the direction AB. The cross-sectional area of the tube is denoted by $\mathscr{A}$. The constant vector $\boldsymbol{\gamma}^A$ is known in principle, and K_1 is a characteristic conductivity of the mouth A (cf. Rayleigh 1945, ch. 16). Similarly $\phi_2^*(\mathbf{y})$ is a potential function normalized such that

$$\phi_2^*(\mathbf{y}) \to \left\{\begin{array}{ll} \mathscr{A}/4\pi r_A & \text{in free space,}\\ y_1+\mathscr{A}/K_2 & \text{within the tube.}\end{array}\right\} \tag{10.11}$$

This function describes irrotational incompressible flow *into* the tube at A, and K_2 is the Rayleigh conductivity of the mouth A.

Our objective is a description of the properties of the Green's function $\mathscr{G}$ in the neighbourhood of A, and in principle this is provided by (10.9) once the parameters α and ϕ_0^A have been determined.

At points in the exterior fluid located several mouth diameters from A, equation (10.9) reduces to

$$\mathscr{G} \simeq Z_A\left\{1 - i\frac{\kappa\mathbf{x}}{|\mathbf{x}|}\cdot\left[\mathbf{y} + O\left(\frac{1}{r_A^2}\right)\right]\right\} + \frac{\alpha\mathscr{A}}{4\pi r_A} + \phi_0^A. \tag{10.12}$$

It is anticipated that $\alpha = O(Z_A l/\mathscr{A})$ and corresponds to the component of $\mathscr{G}$ associated with the resonant modes in the tube.

Since (10.8) and (10.12) are alternative representations of the field in the region of space considered, it follows that

$$i\kappa\phi_A = \phi_0^A, \quad \phi_A = \alpha\mathscr{A}/4\pi,$$

so that

$$\phi_0^A = i\kappa\alpha\mathscr{A}/4\pi. \tag{10.13}$$

Consider next the fluid motion within the tube. In the body of the tube this has the form of the standing wave

$$\mathscr{G} = C\,e^{i\kappa y_1} + D\,e^{-i\kappa y_1}, \tag{10.14}$$

say. Near the end A this is approximately given by

$$\mathscr{G} \simeq (C+D) + i\kappa y_1(C-D) + \ldots, \tag{10.15}$$

which must be equivalent to the corresponding terms in the asymptotic form of (10.9) within the tube, viz.

$$\mathscr{G} \simeq Z_A\left\{1 - \frac{i\kappa\mathbf{x}\,.\,\boldsymbol{\gamma}^A}{|\mathbf{x}|}\left(y_1 + \frac{\mathscr{A}}{K_1}\right)\right\} + \alpha\left(y_1 + \frac{\mathscr{A}}{K_2}\right) + \phi_0^A. \tag{10.16}$$

Hence

$$\left.\begin{aligned} Z_A\left\{1 - i\frac{\kappa\mathbf{x}\,.\,\boldsymbol{\gamma}^A\mathscr{A}}{K_1|\mathbf{x}|}\right\} + \frac{\alpha\mathscr{A}}{K_2} + \phi_0^A &= C+D, \\ \alpha - i\frac{\kappa\mathbf{x}\,.\,\boldsymbol{\gamma}^A}{|\mathbf{x}|}Z_A &= i\kappa(C-D). \end{aligned}\right\} \tag{10.17}$$

To complete the analysis determining α and ϕ_0^A it is necessary to supplement (10.13) and (10.17) by repeating the above matching procedure at the end B of the tube. To do this we express the acoustic field in the exterior fluid at B in the form

$$\begin{aligned} \mathscr{G} &\simeq Z_B\{1 - i\kappa\mathbf{x}\,.\,(\mathbf{y}-\mathbf{l})/|\mathbf{x}|\} + \phi_B(t - r_B/c)/r_B \\ &\simeq Z_B\{1 - i\kappa\mathbf{x}\,.\,(\mathbf{y}-\mathbf{l})/|\mathbf{x}|\} + i\kappa\phi_B + \phi_B/r_B, \end{aligned} \tag{10.18}$$

where $\phi_B \equiv \phi_B(t)$ and $r_B = |\mathbf{y}-\mathbf{l}|$.

The flow in the vicinity of the open end B is specified approximately by the potential

$$\mathscr{G} \simeq Z_B\left\{1 - i\frac{\kappa\mathbf{x}}{|\mathbf{x}|}\cdot[\mathbf{y} - \mathbf{l} + \boldsymbol{\phi}_3^*(\mathbf{y})]\right\} + \beta\phi_4^*(\mathbf{y}) + \phi_0^B(t), \tag{10.19}$$

where $\boldsymbol{\phi}_3^*(\mathbf{y})$ and $\phi_4^*(\mathbf{y})$ are harmonic functions analogous to $\boldsymbol{\phi}_1^*(\mathbf{y})$ and $\phi_2^*(\mathbf{y})$ with asymptotic expressions of the form

$$\mathbf{y}-\mathbf{l}+\boldsymbol{\phi}_3^*(\mathbf{y})\rightarrow\begin{cases}\mathbf{y}-\mathbf{l}+O(r_B^{-2}) & \text{in free space,}\\ \boldsymbol{\gamma}^B(y_1-l-\mathscr{A}/K_3) & \text{within the tube}\end{cases} \tag{10.20}$$

and

$$\phi_4^*(\mathbf{y})\rightarrow\begin{cases}-\mathscr{A}/4\pi r_B & \text{in free space,}\\ y_1-l-\mathscr{A}/K_4 & \text{within the tube.}\end{cases} \tag{10.21}$$

The time-dependent parameters β and ϕ_0^B are to be determined from the analysis.

Carrying through the appropriate matching procedure at B as described above then leads to the following system of consistency equations:

$$\left.\begin{aligned}\phi_0^B &= -i\kappa\beta\mathscr{A}/4\pi,\\ Z_B\left\{1+i\frac{\kappa\mathbf{x}.\boldsymbol{\gamma}^B\mathscr{A}}{K_3|\mathbf{x}|}\right\}-\frac{\beta\mathscr{A}}{K_4}+\phi_0^B &= C\,e^{i\kappa l}+D\,e^{-i\kappa l},\\ \beta-i\kappa\frac{\mathbf{x}.\boldsymbol{\gamma}^B}{|\mathbf{x}|}Z_B &= i\kappa\{C\,e^{i\kappa l}-D\,e^{-i\kappa l}\}.\end{aligned}\right\} \tag{10.22}$$

The procedure now consists of calculating α and ϕ_0^A from the six simultaneous equations (10.13), (10.17) and (10.22). The analysis is tedious but straightforward, and will not be reproduced here. Actually it is considerably simplified without any real loss in generality if all of the terms involving the conductivities K_1, K_2, K_3 and K_4 are neglected. The principal effect of these terms is associated with the 'end corrections' at A and B, which modify the usual 'organ pipe' resonant frequencies of the tube. The corrections are small provided that the blockages at A and B are not too great. When this approximation is introduced the expression for α is found to be

$$\alpha=\frac{i\kappa\mathbf{x}.\boldsymbol{\gamma}^A}{|\mathbf{x}|}Z_A+\frac{\kappa Z_A\{e^{-i\kappa l\cos\theta}-\cos\kappa l\}}{\sin[\kappa l+i\kappa^2\mathscr{A}/2\pi]}, \tag{10.23}$$

where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{l}$.

Thus the leading terms in the low frequency expansion of $\mathscr{G}$ for points $\mathbf{y}$ located in the vicinity of A are given by

$$\mathscr{G}(\mathbf{x},\mathbf{y};t,\omega)=Z_A\left\{1-\frac{i\omega\mathbf{x}}{c|\mathbf{x}|}.[\mathbf{y}+\boldsymbol{\phi}_1^*(\mathbf{y})-\boldsymbol{\gamma}^A\phi_2^*(\mathbf{y})]\right\}$$
$$+\frac{\left(\dfrac{\omega}{c}\right)Z_A\phi_2^*(\mathbf{y})\left[\exp\left(-i\dfrac{\omega l}{c}\cos\theta\right)-\cos\dfrac{\omega l}{c}\right]}{\sin\left\{\dfrac{\omega l}{c}\left(1+i\dfrac{\omega\mathscr{A}}{2\pi cl}\right)\right\}}. \tag{10.24}$$

Recalling the definition (10.5) of Z_A, the reciprocal theorem implies that (10.24) is also the potential at the far-field point $\mathbf{x}$ due to a harmonic point source at $\mathbf{y}$. Multiplying (10.24) by $(2\pi)^{-1}e^{i\omega\tau}$ and integrating over all real ω then leads to the following expression for the low frequency Green's function for source points $\mathbf{y}$ near the mouth A:

$$G(\mathbf{x},\mathbf{y};t,\tau) \simeq \frac{1}{4\pi|\mathbf{x}|}\delta\left\{t-\tau-\frac{|\mathbf{x}-\mathbf{y}-\boldsymbol{\phi}_1^*(\mathbf{y})+\boldsymbol{\gamma}^A\phi_2^*(\mathbf{y})|}{c}\right\}$$
$$+\frac{c\phi_2^*(\mathbf{y})}{2l^2|\mathbf{x}|}H\left\{t-\tau-\frac{l}{c}-\frac{|\mathbf{x}-\mathbf{l}|}{c}\right\}\sum_{n=1}^{\infty}(-1)^{n+1}n\sin\left\{\frac{n\pi c}{l}\left(t-\tau-\frac{|\mathbf{x}-\mathbf{l}|}{c}\right)\right\}$$
$$\times\exp\left\{-\frac{n^2\pi c\mathscr{A}}{2l^3}\left(t-\tau-\frac{|\mathbf{x}-\mathbf{l}|}{c}\right)\right\}+\frac{c\phi_2^*(\mathbf{y})}{4l^2|\mathbf{x}|}\left\{H\left[t-\tau-\frac{|\mathbf{x}|}{c}\right]+H\left[t-\tau-\frac{2l}{c}-\frac{|\mathbf{x}|}{c}\right]\right\}$$
$$\times\sum_{n=1}^{\infty}n\sin\left\{\frac{n\pi c}{l}\left(t-\tau-\frac{|\mathbf{x}|}{c}\right)\right\}\exp\left\{-\frac{n^2\pi c\mathscr{A}}{2l^3}\left(t-\tau-\frac{|\mathbf{x}|}{c}\right)\right\}. \tag{10.25}$$

The first term on the right of this result is of the same form as the free-space Green's function, but with the source position $\mathbf{y}$ in the argument of the delta function modified by geometrical constraints imposed by the mouth A. This term is non-resonant, and is responsible for transient features of the scattered sound. The remaining terms account for the coupling with the resonant modes within the tube. It is clear from the retarded time dependence that the second term represents the field radiated from the open end B and the third term gives the radiation from A. The latter is actually composed of two components, the second of which arises only after the initial impulse within the tube has been reflected at B and returned to A. Both of the resonant contributions to the Green's function involve the characteristic frequencies of the tube, but unlike the delta-function pulse of the transient term, they are sharp-fronted disturbances with relatively extensive tails determined by the relaxation time $2l^3/\{\pi n^2 c\mathscr{A}\}$ of the particular resonant mode. In the present approximate theory, this decay is due entirely to radiation damping.

The calculation outlined above may now be repeated in order to determine the acoustic field *within* the tube due to a point source located at the mouth A. The details and the approximations involved are very similar to those already described and will not be recorded here. We merely note that the result of such an analysis gives the following approximate expression for the time-harmonic Green's function:

$$\mathscr{G}(\mathbf{x},\mathbf{y},t,\omega) = -\phi_2^*(\mathbf{y})\sin\left\{\frac{\omega}{c}(x_1-l)\right\}e^{-i\omega t}\Big/\mathscr{A}\sin\left\{\frac{\omega l}{c}\left(1+i\frac{\omega\mathscr{A}}{2\pi cl}\right)\right\}, \tag{10.26}$$

where the source point $\mathbf{y}$ is located near A and x_1 is several mouth diameters within the tube from A.

The real-time form of the Green's function is obtained by multiplying (10.26) by $(2\pi)^{-1}e^{i\omega\tau}$ and integrating over all ω, and this gives

$$G(\mathbf{x},\mathbf{y};t,\tau) \equiv -\frac{1}{2\pi}\int\frac{\phi_2^*(\mathbf{y})\sin\{\omega c^{-1}(x_1-l)\}\exp[-i\omega(t-\tau)]\,d\omega}{\mathscr{A}\sin\left\{\dfrac{\omega l}{c}\left(1+\dfrac{i\omega\mathscr{A}}{2\pi cl}\right)\right\}}$$
$$=\frac{c\phi_2^*(\mathbf{y})}{l\mathscr{A}}\sum_{n=1}^{\infty}(-1)^n\left\{H\left(t-\tau-\frac{x_1}{c}\right)\cos\left[\frac{n\pi c}{l}\left(t-\tau-\frac{x_1-l}{c}\right)\right]\right.$$
$$\left.-H\left(t-\tau-\frac{2l-x_1}{c}\right)\cos\left[\frac{n\pi c}{l}\left(t-\tau+\frac{x_1-l}{c}\right)\right]\right\}\exp\left(-\frac{\pi n^2 c\mathscr{A}}{2l^3}(t-\tau)\right). \tag{10.27}$$

This result exhibits the expected features. The first term in the curly brackets in the summation is a resonant wave propagating in the $+x_1$ direction and arises after the shortest possible arrival time $t-\tau = x_1/c$. The second such term represents a mode propagating in the $1-x_1$ direction, and the arrival time

$$t-\tau = (2l-x_1)/c$$

shows that it is initiated by the reflexion of the initial pulse from the end B of the tube. The exponential factor represents the relaxation of wave modes in the tube due to radiation from the open ends. If this is neglected the two Fourier series sum to equal but opposite *constant* values, in which case the wave reflected at B completely annihilates the incident pulse. When both sets of waves are established, i.e. for $t > \tau+(2l-x_1)/c$, (10.27) assumes the standing wave form

$$G(\mathbf{x}, \mathbf{y}; t, \tau) = \frac{2c\phi_2^*(\mathbf{y})}{l\mathscr{A}} \sum_{n=1}^{\infty} (-1)^n \sin\left\{\frac{n\pi c}{l}(t-\tau)\right\} \times \sin\left\{\frac{n\pi}{l}(x_1-l)\right\} \exp\left\{-\frac{\pi n^2 c\mathscr{A}}{2l^3}(t-\tau)\right\}. \quad (10.28)$$

Stage II: the vortex shedding model

The dipole source term $\operatorname{div}(\boldsymbol{\omega} \wedge \mathbf{v})$ in (10.1) must be determined on the basis of an idealized mathematical model, and the procedure we shall adopt is closely related to certain calculations which have been performed in connexion with the formation of leading-edge vortices for slender delta wings (Brown & Michael 1955; Smith 1959). Specifically, it will be assumed that the principal features of the vortex shedding mechanism are contained in the local two-dimensional problem depicted in figure 6.

In this model the mouth A is rectangular with sides of length $2s$ parallel to the tube AB. The transverse dimension of the mouth (into the paper) is of length d. The fluid motion in the vicinity of the mouth is taken to be two-dimensional and in the 1, 2 plane, that of the paper, the local geometry of the tube being approximated by an infinite plane containing a slit of width $2s$. In figure 6 the mean flow U is in the $+x_1$ or $\mathbf{i}$ direction. Superimposed on this, however, is a time-dependent cross-flow produced by the sound field in the interior of the tube, which is assumed to be specified by incompressible potential flow through the slit. Such a flow is singular at the lip O at which the air stream is blown, and in a real fluid the associated large velocity gradients produce a significant viscous effect, the flow actually separating at the edge and resulting in the formation of a spiral vortex sheet which develops in time. We suppose that an adequate representation of this vortex field is obtained by assuming that it rolls up into an intense vortex core (cf. Brown & Michael 1955; Smith 1959).

Two further idealizations are now introduced. The first is that, *as far as the generation of vorticity is concerned*, the cross-flow velocity through the slit may be taken to be *constant in time for each half-cycle* of the acoustic mode under consideration. Thus, if V denotes the average cross-flow velocity in the plane of the slit in such a half-cycle, so that the corresponding volume flux is $2sdV$, then the

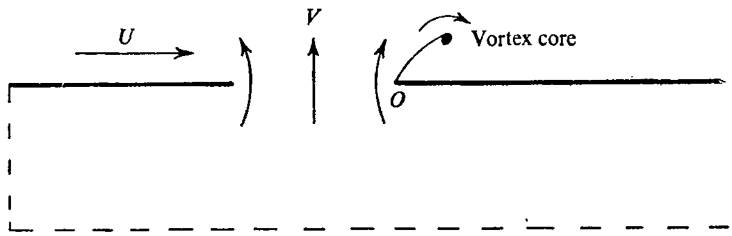

FIGURE 6. Two-dimensional model of vortex shedding. Flow in the mouth A is modelled by the two-dimensional irrotational theory of the flow through a slit in an infinite plane. The mean cross-flow velocity V is assumed to be constant for the purpose of calculating the properties of the vortex core, and reverses direction at the end of each half-cycle of the acoustic oscillation.

complex potential w describing the potential flow through the slit is given by

$$w = \frac{Vs}{\pi}\ln\left\{\left(1+\frac{z}{s}\right)+\left[\left(1+\frac{z}{s}\right)^2-1\right]^{\frac{1}{2}}\right\} \tag{10.29}$$

(Lamb 1932, p. 73), where $z = x_1+ix_2$ and the origin of the co-ordinates is taken at the lip O. Figure 6 illustrates the situation during a half-cycle in which the cross-flow is directed out of the mouth A.

Second, it will be assumed that during the period in which the vortex core is exciting sound within the tube it is located sufficiently close to the lip O that $|z/s| \ll 1$. Then (10.29) becomes approximately

$$w = (V/\pi)(2sz)^{\frac{1}{2}}. \tag{10.30}$$

Thus in the absence of vortex shedding the flow potential near the lip O has the form

$$w_0 = Uz+(V/\pi)(2sz)^{\frac{1}{2}}. \tag{10.31}$$

Let $z_0 = X_1+iX_2 \equiv R\,e^{i\phi}$ be the complex position of the vortex core at time t, and let $\Gamma \equiv \Gamma(t)$ be the vortex strength, then the additional potential due to the vortex which must be added to (10.31) to give the total velocity potential is

$$w_1 = i\Gamma\ln\left\{\frac{z^{\frac{1}{2}}-z_0^{*\frac{1}{2}}}{z^{\frac{1}{2}}-z_0^{\frac{1}{2}}}\right\}. \tag{10.32}$$

Combining (10.31) and (10.32), the vortex strength Γ is now chosen to ensure that the flow velocity remains finite at the lip O (Kutta condition), from which it follows that

$$\Gamma = -\frac{V(2s)^{\frac{1}{2}}}{2\pi}\frac{R^{\frac{1}{2}}}{\sin\frac{1}{2}\phi}. \tag{10.33}$$

Reference to figure 6 reveals that the minus sign in (10.33) is consistent with the expected direction of the rotational flow.

The motion of the vortex is determined by subtracting the vortical self-potential $-i\Gamma\ln(z-z_0)$ from w_0+w_1, the resulting expression being the velocity potential for the core motion. This leads to the following pair of equations for the vortex path:

$$\left.\begin{aligned}\frac{dX_1}{dt} &= U-\frac{V}{8\pi}\left(\frac{2s}{R}\right)^{\frac{1}{2}}\frac{\cos\frac{1}{2}\phi\cos\phi}{\sin^2\frac{1}{2}\phi},\\ \frac{dX_2}{dt} &= \frac{V}{4\pi}\left(\frac{2s}{R}\right)^{\frac{1}{2}}\sin\frac{1}{2}\phi.\end{aligned}\right\} \tag{10.34}$$

Consider the situation during a half-cycle in which the cross-flow is directed *out* of the mouth A. At the beginning of this half-cycle, at $t = 0$, say, the flow velocity jumps to the constant value $+V$; the flow subsequently changes sign discontinuously at the end of this first half-cycle, at which stage the vortex illustrated in figure 6 has grown to its full strength and is released. A new vortex then forms *within* the tube during the second half-cycle. It is easy to show that (10.34) admit solutions of the form

$$\left.\begin{aligned} X_1 &= at + O(t^{\frac{4}{3}}), \\ X_2 &= b_1 t^{\frac{2}{3}} + b_2 t + O(t^{\frac{4}{3}}), \end{aligned}\right\} \tag{10.35}$$

in which the initial trajectory of the vortex core is in the x_2 direction. We shall assume that the approximate terms shown explicitly in (10.35) give an adequate characterization of the vortex path during the occurrence of the dominant acoustic excitation. Substituting (10.35) into (10.34) then gives

$$X_1 \simeq \tfrac{3}{5}Ut, \quad X_2 \simeq (\mu t)^{\frac{2}{3}} - \tfrac{3}{20}Ut, \tag{10.36}$$

where

$$\mu = (3V/8\pi)\, s^{\frac{1}{2}}. \tag{10.37}$$

If, as before, the three unit vectors (**i**, **j**, **k**) are directed respectively in the (1, 2, 3) directions, then the velocity of translation of the vortex at a time t after the beginning of the first half-cycle is

$$\mathbf{v} \simeq \tfrac{3}{5}U\mathbf{i} + \{\tfrac{2}{3}\mu^{\frac{2}{3}}t^{-\frac{1}{3}} - \tfrac{3}{20}U\}\,\mathbf{j}. \tag{10.38}$$

The vorticity vector $\boldsymbol{\omega}$ is effectively non-zero only within a distance d (the transverse dimension of the mouth A) in the x_3 direction, where it is given by

$$\boldsymbol{\omega} = 2\pi\Gamma\mathbf{k}\,\delta[x_1 - X_1(t)]\,\delta[x_2 - X_2(t)]. \tag{10.39}$$

This completes the specification of $\mathbf{v}$ and $\boldsymbol{\omega}$.

Stage III: the acoustic response of the flute

The solution of the acoustic-analogy equation (10.1) for the field in free space or within the tube is obtained by convoluting the dipole source term with the appropriate Green's function. In the case of excitation by periodic vortex shedding at the lip O, the spatial part of the convolution involves the integral

$$\begin{aligned} I(t) &= \int \operatorname{div}(\boldsymbol{\omega} \wedge \mathbf{v})\, \phi_2^*(\mathbf{y})\, d^3\mathbf{y} \\ &= -\int \boldsymbol{\omega} \wedge \mathbf{v} \,.\, \nabla\phi_2^*\, d^3\mathbf{y}. \end{aligned} \tag{10.40}$$

Using the result (10.39) this becomes

$$I(t) = -2\pi d[\Gamma\mathbf{k} \wedge \mathbf{v} \,.\, \nabla\phi_2^*], \tag{10.41}$$

where the quantity in the square brackets is evaluated at the position of the vortex at time t.

Now the potential function $\phi_2^*(\mathbf{y})$ is normalized in such a manner that it represents irrotational flow *into* the mouth A with a total flux $\mathscr{A}$ [cf. definition

(10.11)]. In the vicinity of the mouth $\phi_2^*(\mathbf{y})$ must have a functional form proportional to the real part of the right-hand side of (10.29). The constant of proportionality is chosen to make the flux through the mouth equal to $\mathscr{A}$, from which it follows that, near the lip O,

$$\phi_2^*(\mathbf{y}) = \operatorname{Re}\left\{-\frac{\mathscr{A}}{\pi d}\left(\frac{z}{2s}\right)^{\frac{1}{2}}\right\}. \tag{10.42}$$

Hence, using formulae (10.33) and (10.38) we deduce that in terms of the polar co-ordinates (R, ϕ) of the vortex core at a time t after the beginning of the first half-cycle (cross-flow directed *out* of the mouth A)

$$I(t) = -\frac{V\mathscr{A}}{2\pi}\left\{\left(\frac{3U}{20}-\tfrac{2}{3}\mu^{\frac{2}{3}}t^{-\frac{1}{3}}\right)\cot\frac{\phi}{2}+\frac{3U}{5}\right\}. \tag{10.43}$$

Now $\cot\tfrac{1}{2}\phi = 1+\tfrac{3}{5}U\mu^{-\frac{2}{3}}t^{\frac{1}{3}}+\dots$, so that we can also write approximately

$$I(t) = -\frac{V\mathscr{A}}{2\pi}\left\{\frac{7U}{20}-\tfrac{2}{3}\mu^{\frac{2}{3}}t^{-\frac{1}{3}}\right\}. \tag{10.44}$$

If the frequency of the acoustic oscillation under consideration is equal to $\omega_n = n\pi c/l$, then this expression gives $I(t)$ in the range $0 < t < \pi/\omega_n$. During the second half-cycle, when the cross-flow is directed *into* the mouth A, a similar argument shows that

$$I(t) = +\frac{V\mathscr{A}}{2\pi}\left\{\frac{7U}{20}-\tfrac{2}{3}\mu^{\frac{2}{3}}(t-\pi/\omega_n)^{-\frac{1}{3}}\right\}. \tag{10.45}$$

Equation (10.44), which gives the effective dipole source strength $I(t)$ during the first half-cycle, actually represents only the leading terms in the expansion about $t = 0$ (it may be verified *a posteriori* that the dimensionless expansion parameter is proportional to $(ct/l)^{\frac{1}{3}}$). In spite of this, the result already exhibits the competing influences of the mean flow and the cross-flow, represented respectively by the first and second terms in the curly brackets in (10.44), on the excitation of sound in the tube. In order to be consistent with the approximations already made, the function $I(t)$ must be regarded as an *effective* source in much the same way that the step-wise behaviour of the cross-flow velocity V is an approximate representation of a sinusoidally varying flow in the mouth A. The time dependence of $I(t)$ during the first half-cycle must actually be in phase with the cross-flow through the mouth, i.e. of the form

$$I(t) = \sum_N I_N \sin N\omega_n t \quad (0 < t < \pi/\omega_n).$$

Equation (10.44) does not furnish sufficient information to determine the coefficients of this expansion. However, on the basis that most of the acoustic excitation occurs at the *beginning* of the cycle, it is reasonable to suppose that (10.44) provides an adequate description for obtaining at least a *first approximation* to the amplitude I_1 of the fundamental term in the sine series.

This is given by

$$I_1 = \frac{\omega_n}{2\pi}\int_0^{\pi/\omega_n} I(t)\sin\omega_n t\,dt, \tag{10.46}$$

i.e. using (10.44) we have the approximate result

$$I_1 = -\frac{V\mathscr{A}}{\pi^2}\left\{\frac{7U}{10}-\frac{2J}{3}(\mu\omega_n^{\frac{1}{2}})^{\frac{2}{3}}\right\}, \tag{10.47}$$

where $$J = \int_0^\pi \frac{\sin z\, dz}{z^{\frac{1}{3}}} \simeq 1{\cdot}83.$$

It is easy to see that the formula $I(t) \simeq I_1 \sin\omega_n t$, with I_1 determined by (10.47), can be continued into the second half-cycle ($\pi/\omega_n < t < 2\pi/\omega_n$), since it also represents the corresponding approximation to (10.45). In other words the principal effect of periodic vortex shedding is embodied in the single formula $I(t) = I_1 \sin\omega_n t$.

Taking account of the definition (10.40) of $I(t)$, it follows from this and the interior Green's function (10.27) that the periodic acoustic response within the tube is given approximately by the convolution integral

$$B = -\frac{1}{2\pi}\iint \frac{I_1 \sin\omega_n\tau \sin\{\omega c^{-1}(x_1-l)\}\exp\{-i\omega(t-\tau)\}\,d\omega\, d\tau}{\mathscr{A}\sin\left\{\dfrac{\omega l}{c}\left(1+\dfrac{i\omega\mathscr{A}}{2\pi c l}\right)\right\}}$$

$$= \frac{2I_1 l^2}{n^2\pi\mathscr{A}^2}(-1)^{n+1}\sin\left\{\frac{n\pi}{l}(x_1-l)\right\}\cos\frac{n\pi ct}{l}. \tag{10.48}$$

Now the flow velocity u_1 in the tube is related to the stagnation enthalpy by $\partial u_1/\partial t = -\partial B/\partial x_1$. Thus (10.48) implies that near the mouth A ($x_1/l \ll 1$) the flow velocity is given by

$$u_1 = \frac{2I_1 l^2}{n^2\pi c\mathscr{A}^2}\sin\frac{n\pi ct}{l}, \tag{10.49}$$

a result which is seen to be *in phase* with the cross-flow velocity V, as required. During the half-cycle $0 < t < \pi/\omega_n$ ($\omega_n = n\pi c/l$), this must give rise to a flux *out* of the mouth A which is precisely equal to that represented by the constant velocity V:

$$\frac{2sdV\pi}{\omega_n} = \int_0^{\pi/\omega_n} -\mathscr{A}u_1(t)\,dt, \tag{10.50}$$

the right-hand side being the net flux in the $-x_1$ direction through a cross-section of the tube located several mouth diameters from A. It follows that

$$I_1 = -\tfrac{1}{2}n^2\pi^2\mathscr{A}(sd/l^2)cV. \tag{10.51}$$

Substituting for I_1 from (10.47), and recalling that $\mu = 3Vs^{\frac{1}{2}}/8\pi$, we finally obtain the following expression for the magnitude V of the cross-flow velocity:

$$V = c\frac{8}{3}\left(\frac{21}{20J}\right)^{\frac{3}{2}}\left(\frac{\pi l}{ns}\right)^{\frac{1}{2}}\left\{M-\tfrac{5}{7}n^2\pi^4\left(\frac{sd}{l^2}\right)\right\}^{\frac{3}{2}}, \tag{10.52}$$

where $M = U/c$. This formula determines the effective cross-flow velocity in terms of the dimensions s and d of the mouth A, the length l of the tube and the blowing Mach number M of the incident air stream. It is open to the following interpretation.

It has been assumed that the frequency of the cross-flow velocity is equal to that of one of the resonant standing modes n. Such a standing wave can exist within the tube provided that the energy supplied by the incident air stream is sufficient to overcome the dissipation in the system. Equation (10.52) indicates that this requires

$$M > \tfrac{5}{7}n^2\pi^4(sd/l^2), \tag{10.53}$$

in which case the cross-flow velocity is *real*. The quantity on the right of this inequality is proportional to the rate at which energy is lost from the openings in the tube at A and B in the form of acoustic radiation [cf. the decay factors in (10.27)]. The threshold Mach number predicted by (10.53) for a particular value of n is therefore necessarily a *minimum*, since we have neglected the dissipation associated with viscous and heat-conduction losses at the walls of the tube. The inclusion of such effects would lead to the additive presence of further positive-definite terms on the right of (10.53).

When the condition (10.53) is satisfied by several values of n it is necessary to decide which mode actually sounds. Our analysis is too crude to include this choice, but it is reasonable to assume that the tube sounds in that mode of oscillation which has *minimum* energy at the given incident stream Mach number M. Since the wave energy within the tube is proportional to V^2, it is clear that the minimum energy mode will correspond to the *largest* value of n which satisfies the inequality $n < \pi^{-2}(7Ml^2/5sd)^{\frac{1}{2}}$. This provides a law governing the excitation of the higher-order modes with increasing blowing pressure.

A transition to one of the higher-order modes can also be effected by an alternative procedure. Instead of increasing the blowing Mach number M, the effective width $2s$ of the mouth is *decreased*. It is interesting to note that this is actually part of the playing technique used by the flautist in controlling the octave in which his instrument sounds (Coltman 1968).

The acoustic field radiated into free space from the ends of the tube is obtained by evaluating a convolution integral similar to (10.48) above, but using the resonant part of the free-space Green's function (10.24). This leads to the following expression for the distant pressure field:

$$p \simeq \rho_0 c^2 \frac{8}{3}\left(\frac{21}{20J}\right)^{\frac{3}{2}}\left(\frac{d}{|\mathbf{x}|}\right)\left(\frac{\pi^3 ns}{l}\right)^{\frac{1}{2}}\left[M - \tfrac{5}{7}n^2\pi^4\left(\frac{sd}{l^2}\right)\right]^{\frac{3}{2}}$$
$$\times\left\{\cos\left[\frac{n\pi c}{l}\left(t-\frac{|\mathbf{x}|}{c}\right)\right]-(-1)^n\cos\left[\frac{n\pi c}{l}\left(t-\frac{|\mathbf{x}-\mathbf{l}|}{c}\right)\right]\right\}. \tag{10.54}$$

The terms in the curly brackets correspond respectively to the sound radiated from the openings in the tube at A and B.

Our results may be illustrated numerically by the case of a *descant* recorder, which has the approximate dimensions $s = 0{\cdot}2$ cm, $d = 0{\cdot}9$ cm, $l = 28$ cm. The minimum threshold blowing Mach number M for the excitation of the fundamental frequency ($\sim$ 500 Hz) is obtained by setting $n = 1$ on the right of (10.53), and corresponds to a blowing velocity of about 5 m s^{-1}. When the dissipation due to viscous and heat-conduction effects is taken into account, the actual blowing velocity must be in excess of this. However the intensity of the radiated sound is dependent on the details of the dissipation mechanisms only through the factor $[M-\frac{5}{7}n^2\pi^4(sd/l^2)]^{\frac{3}{2}}$ in (10.54). If it is assumed that the blowing velocity exceeds the actual threshold velocity by $0{\cdot}5$ m s^{-1}, then (10.54) indicates that at a distance of about 1 m from the mouth of the recorder the overall sound pressure level is about 66 dB; for an excess blowing velocity of 1 m s^{-1} the corresponding level is 75 dB. Both of these predictions are in excellent order-of-magnitude agreement with measurements made by the author.

Note added in proof. Some very recent experimental results obtained by Fletcher reveal that the sounding frequency of the flute is proportional to the blowing pressure, i.e. $n \propto M^2$. If, in the above analysis, account is taken of boundary-layer dissipation, (10.53) becomes

$$M > \frac{5\pi^4}{7}\frac{sd}{\mathscr{A}}\left\{\frac{n^2\mathscr{A}}{l^2} + \left(\frac{2nl}{c\mathscr{A}}\right)^{\frac{1}{2}}[\nu^{\frac{1}{2}} + \chi^{\frac{1}{2}}(\gamma-1)]\right\},$$

where ν and χ are respectively the kinematic viscosity and the thermometric conductivity, The second term in the curly brackets greatly exceeds the first (which represents radiation damping), and Fletcher's result may then be deduced by applying the minimum-energy argument.

The author acknowledges the benefit he has derived from discussions of the material of this paper with his colleagues at the Cambridge Noise Research Unit, and in particular with Professor J. E. Ffowcs Williams. The work was supported by the Bristol Engine Division of Rolls Royce (1971) Ltd.

Appendix. Low frequency Green's functions

Let $\mathscr{L}(\partial/\partial \mathbf{x}, \partial/\partial t)$ be a linear acoustic wave operator of the type appearing in the main text. The Green's function for this operator is defined to be the particular solution of the equation

$$\mathscr{L}(\partial/\partial \mathbf{x}, \partial/\partial t)\, G(\mathbf{x}, \mathbf{y}; t, \tau) = \delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau) \tag{A 1}$$

which satisfies the radiation condition at large distances, $|\mathbf{x}| \to \infty$, and the condition of vanishing normal derivative, $\partial G/\partial n = 0$, on any rigid surfaces in the flow.

When G is known the causal solution of the problem involving an arbitrary source distribution, i.e. of

$$\mathscr{L}B = f(\mathbf{x}, t), \tag{A 2}$$

is given by the convolution integral

$$B(\mathbf{x}, t) = \int f(\mathbf{y}, \tau)\, G(\mathbf{x}, \mathbf{y}; t, \tau)\, d^3\mathbf{y}\, d\tau. \tag{A 3}$$

For acoustic waves propagating in free space we have

$$\left.\begin{aligned} \mathscr{L} &\equiv c^{-2}\,\partial^2/\partial t^2 - \nabla^2, \\ G(\mathbf{x}, \mathbf{y}; t, \tau) &= \frac{1}{4\pi}\,\frac{\delta\{t-\tau-|\mathbf{x}-\mathbf{y}|/c\}}{|\mathbf{x}-\mathbf{y}|}. \end{aligned}\right\} \tag{A 4}$$

Introduce the Fourier time transform $\bar{G}(\mathbf{x}, \mathbf{y}, \omega)$ defined by

$$G(\mathbf{x}, \mathbf{y}; t, \tau) = \int \bar{G}(\mathbf{x}, \mathbf{y}, \omega) \exp\{-i\omega(t-\tau)\}\, d\omega, \tag{A 5}$$

and suppose that $\bar{G}(\mathbf{x}, \mathbf{y}, \omega)$ can be developed into an asymptotic series of the form

$$\bar{G}(\mathbf{x}, \mathbf{y}, \omega) = \sum_{n\geqslant 0} i^n \omega^n a_n(\mathbf{x}, \mathbf{y}) \exp\{i\omega b_n(\mathbf{x}, \mathbf{y})\}. \tag{A 6}$$

In practice the terms in this expansion diminish rapidly with increasing n provided that the source location $\mathbf{y}$ is restricted to a suitable compact region of space.

Equations (A 5) and (A 6) imply the formal equivalence

$$G(\mathbf{x}, \mathbf{y}; t, \tau) = 2\pi \sum_{n\geqslant 0} (-1)^n a_n(\mathbf{x}, \mathbf{y})\, \delta^{(n)}\{t-\tau-b_n(\mathbf{x}, \mathbf{y})\}, \tag{A 7}$$

so that the convolution integral (A 3) becomes

$$B(\mathbf{x}, t) = 2\pi \sum_{n\geqslant 0} \int a_n(\mathbf{x}, \mathbf{y}) \frac{\partial^n f}{\partial t^n}(\mathbf{y}, t-b_n(\mathbf{x}, \mathbf{y}))\, d^3\mathbf{y}. \tag{A 8}$$

This result expresses the radiation field as an asymptotic expansion involving successive time derivatives of the source function. Provided that the characteristic source frequency is sufficiently small only the first few terms are significant. In the approximation which takes into account only the first *two* terms, (A 7) becomes

$$G(\mathbf{x}, \mathbf{y}; t, \tau) \simeq 2\pi\{a_0(\mathbf{x}, \mathbf{y})\, \delta[t-\tau-b_0(\mathbf{x}, \mathbf{y})] - a_1(\mathbf{x}, \mathbf{y})\, \delta^{(1)}[t-\tau-b_1(\mathbf{x}, \mathbf{y})]\}. \tag{A 9}$$

This result may be expressed in the alternative approximate form

$$G(\mathbf{x}, \mathbf{y}; t, \tau) \simeq 2\pi a_0(\mathbf{x}, \mathbf{y})\, \delta\left[t-\tau-\left\{\frac{a_0(\mathbf{x}, \mathbf{y})\, b_0(\mathbf{x}, \mathbf{y}) + a_1(\mathbf{x}, \mathbf{y})\, b_1(\mathbf{x}, \mathbf{y})}{a_0(\mathbf{x}, \mathbf{y})}\right\}\right], \tag{A 10}$$

and this is defined as the *low frequency* Green's function of the problem. It is often more convenient than the separated form (A 9) because it is usually possible to renormalize the terms in the curly brackets in (A 10) in such a manner that the formula is valid for an *arbitrary* source location $\mathbf{y}$.

The determination of the approximation (A 10) is sometimes facilitated by means of the *reciprocal theorem*. The following form of the theorem is appropriate to problems discussed in this paper. Let $\phi_A(\mathbf{x}, t)$ be the solution of the convected wave equation

$$\left\{\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)^2 - \nabla^2\right\}\phi_A = \delta(\mathbf{x}-\mathbf{x}_A)\, e^{-i\omega t} \tag{A 11}$$

which satisfies the radiation condition and has vanishing normal derivative on any rigid surfaces in the flow, where the convection velocity $\mathbf{U}$ describes a time-independent, incompressible, irrotational mean flow compatible with the conditions of the problem.

Let ϕ_B be the corresponding solution of the *reverse flow* problem, in which the direction of $\mathbf{U}$ is reversed at all points:

$$\left\{\frac{1}{c^2}\left(\frac{\partial}{\partial t} - \mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)^2 - \nabla^2\right\}\phi_B = \delta(\mathbf{x}-\mathbf{x}_B)\, e^{-i\omega t}. \tag{A 12}$$

Then the reciprocal theorem asserts that $\phi_A(\mathbf{x}_B, t) \equiv \phi_B(\mathbf{x}_A, t)$ (cf. Howe 1975). In particular the result is true when $\mathbf{U}$ vanishes.

Howe (1975) has used the theorem to demonstrate that, for the case of low Mach number irrotational mean flow past an acoustically compact solid depicted in figure 3, the low frequency Green's function (A 10) is given by

$$G(\mathbf{x},\mathbf{y};t,\tau) \simeq \frac{1}{4\pi|\mathbf{X}-\mathbf{Y}|}\,\delta\left\{t-\tau-\frac{|\mathbf{X}-\mathbf{Y}|}{c}+\frac{\mathbf{M}_0.(\mathbf{X}-\mathbf{Y})}{c}\right\}. \qquad (A\ 13)$$

In this formula $\mathbf{X}$ is related to the space co-ordinate $\mathbf{x}$ by

$$X_i = x_i + \phi_i^*(\mathbf{x}), \qquad (A\ 14)$$

and defines the potential of an incompressible irrotational flow past the solid whose velocity at large distances is of unit magnitude and in the i direction. Equation (A 13) describes the generation of sound for an *arbitrary* source position $\mathbf{y}$ and observation point $\mathbf{x}$ provided that at least one of these points is further than an acoustic wavelength from the solid.

For a spherical solid of radius R,

$$X_i = x_i(1+R^3/2|\mathbf{x}|^3), \qquad (A\ 15)$$

where the origin of the co-ordinates is at the centre of the sphere.

In the case of the duct problem illustrated in figure 2, and for an acoustic source located at $\mathbf{y}$, well within a characteristic wavelength from the contraction, the low frequency Green's function can also be determined by means of the reciprocal theorem. However, the leading term in the asymptotic expansion (A 6) is now $O(\omega^{-1})$, so that the corresponding low frequency Green's function for an observation point $\mathbf{x}$ located many wavelengths *downstream* of the contraction has the form

$$G(\mathbf{x},\mathbf{y};t,\tau) = \frac{c}{A_1+A_2}H\left\{t-\tau-\frac{x_1}{c(1+M_2)}+\frac{A_1}{A_2}\frac{\phi^*(\mathbf{y})}{c(1+M_1)}\right\}, \qquad (A\ 16)$$

where $M_1 = U_1/c$, $M_2 = U_2/c$ and $\phi^*(\mathbf{x})$ is a harmonic function describing irrotational flow through the duct and normalized such that $\phi^* \to x_1$ as $x_1 \to \infty$. (See Ffowcs Williams & Howe 1975.)

Low frequency Green's function for a rigid half-plane

We now outline the derivation of (5.14). Consider a harmonic point source located at $\mathbf{y}$ well within an acoustic wavelength of the edge of a rigid plane which occupies $x_2 = 0$, $x_1 < 0$. It is required to solve

$$(c^{-2}\,\partial^2/\partial t^2 - \nabla^2)\,\bar{G} = \delta(\mathbf{x}-\mathbf{y})\,e^{-i\omega t}, \qquad (A\ 17)$$

with $\partial\bar{G}/\partial x_2 = 0$ on the half-plane. By the reciprocal theorem source and observer may be interchanged, so that the source on the right of (A 17) may be regarded as situated at a point $\mathbf{x}$ many wavelengths from the edge of the plane, and the problem reduces to determining $\bar{G}$ as a function of positions $\mathbf{y}$ close to the edge. This is a particular case of the classical Sommerfeld diffraction problem and can be solved in the manner described, for example, by Crighton & Leppington (1970). In that paper it is shown that near the edge of the plate

$$\bar{G} = \bar{G}_0 + \bar{G}_s, \qquad (A\ 18)$$

where
$$\left.\begin{aligned}\bar{G}_0 &= \frac{1}{4\pi|\mathbf{x}-y_3\mathbf{k}|}\exp\left\{-i\omega\left(t-\frac{|\mathbf{x}-y_3\mathbf{k}|}{c}\right)\right\},\\ \bar{G}_s &= \frac{-i}{\pi(2\pi)^{\frac{1}{2}}}\frac{\phi^*(\mathbf{x})\phi^*(\mathbf{y})}{|\mathbf{x}-y_3\mathbf{k}|^{\frac{3}{2}}}\left(\frac{\omega}{c}\right)^{\frac{1}{2}}\exp\left\{i\left[\frac{\pi}{4}-\omega\left(t-\frac{|\mathbf{x}-y_3\mathbf{k}|}{c}\right)\right]\right\}.\end{aligned}\right\} \quad \text{(A 19)}$$

In this result $\mathbf{k}$ is a unit vector parallel to the edge of the half-plane and to the x_3 axis, and
$$\phi^*(\mathbf{x}) = R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta, \quad \text{(A 20)}$$
which is the harmonic function describing irrotational flow about the half-plane expressed in terms of polar co-ordinates defined by $(x_1, x_2) = R(\cos\theta, \sin\theta)$.

Reverting to the original problem, in which the source is at $\mathbf{y}$, the contribution $\bar{G}_0$ represents the field due to a point source in the absence of scattering. The second term $\bar{G}_s$ describes the leading approximation to the *scattered* field. In applications to two-dimensional problems of the type discussed in § 5, the source distribution $f(\mathbf{x}, t)$ has no x_3 dependence, and the appropriate form for the Green's function is obtained by integrating the expressions in (A 19) over all values of y_3. Since $(\omega/c)\,|\mathbf{x}-y_3\mathbf{k}|$ is large this is easily done by the method of stationary phase.

Thus for the scattered field we have
$$\begin{aligned}\int_{-\infty}^{\infty}\bar{G}_s dy_3 &= \frac{-i\phi^*(\mathbf{x})\phi^*(\mathbf{y})}{\pi(2\pi)^{\frac{1}{2}}}\left(\frac{\omega}{c}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{\exp\left\{i\left[\frac{\pi}{4}-\omega\left(t-\frac{|\mathbf{x}-y_3\mathbf{k}|}{c}\right)\right]\right\}}{|\mathbf{x}-y_3\mathbf{k}|^{\frac{3}{2}}}dy_3\\ &\simeq \frac{\phi^*(\mathbf{x})\phi^*(\mathbf{y})}{\pi|\mathbf{x}|}\exp\{-i\omega(t-|\mathbf{x}|/c)\},\end{aligned} \quad \text{(A 21)}$$
where $|\mathbf{x}| = (x_1^2+x_2^2)^{\frac{1}{2}}$ is the distance of the observation point from the edge of the plane. Multiply this result by $(2\pi)^{-1}e^{i\omega\tau}$ and integrate over all ω to obtain the following low frequency approximation to the scattering Green's function:
$$G_s(\mathbf{x}, \mathbf{y}; t, \tau) \simeq \frac{\phi^*(\mathbf{x})\phi^*(\mathbf{y})}{\pi|\mathbf{x}|}\delta\left\{t-\tau-\frac{|\mathbf{x}|}{c}\right\}, \quad \text{(A 22)}$$
where $\mathbf{x}$ and $\mathbf{y}$ denote position vectors in the 1, 2 plane.

The corresponding contribution to the two-dimensional Green's function from $\bar{G}_0$ is independent of $\mathbf{y}$ and is therefore of significance only in applications to problems involving *monopole* source distributions.

REFERENCES

Atvars, J., Schubert, L. K. & Ribner, H. S. 1965 *J. Acoust. Soc. Am.* **37**, 168.

Backus, J. 1970 *The Acoustical Foundations of Music*. London: John Murray.

Blokhintsev, D. I. 1946 Acoustics of a nonhomogeneous moving medium. *N.A.C.A. Tech. Memo.* no. 1399.

Brown, C. E. & Michael, W. M. 1955 On slender delta wings with leading edge separation. *N.A.C.A. Tech. Note*, no. 3430.

Brown, G. B. 1937 *Proc. Phys. Soc. Lond.* **49**, 493.

Candel, S. M. 1972 Analytical studies of some acoustic problems of jet engines. Ph.D. thesis, Caltech.

Cannell, P. & Ffowcs Williams, J. E. 1973 *J. Fluid Mech.* **58**, 65.

CHERNOV, L. A. 1960 *Wave Propagation in a Random Medium.* McGraw Hill.

COLTMAN, J. W. 1968 *Phys. Today*, **21** (11), 25.

CRIGHTON, D. G. 1972 *J. Fluid Mech.* **51**, 357.

CRIGHTON, D. G. & LEPPINGTON, F. G. 1970 *J. Fluid Mech.* **43**, 721.

CRIGHTON, D. G. & LESSER, M. B. 1974 Physical acoustics and the method of matched asymptotic expansions. In *Physical Acoustics* (ed. R. N. Thurston & W. P. Mason), vol. 11, p. 75. Academic.

CROW, S. C. 1967 *Phys. Fluids*, **10**, 1587.

CROW, S. C. 1970 *Studies in Appl. Math.* **49**, 21.

CURLE, N. 1953 *Proc. Roy. Soc.* A **216**, 412.

DOAK, P. E. 1973 *J. Sound Vib.* **26**, 91.

FFOWCS WILLIAMS, J. E. 1963 *Phil. Trans.* A **255**, 469.

FFOWCS WILLIAMS, J. E. 1969 *Ann. Rev. Fluid. Mech.* **1**, 197.

FFOWCS WILLIAMS, J. E. & HALL, L. H. 1970 *J. Fluid Mech.* **40**, 657.

FFOWCS WILLIAMS, J. E. & HAWKINGS, D. L. 1969 *Phil. Trans.* A **264**, 321.

FFOWCS WILLIAMS, J. E. & HOWE, M. S. 1975 *J. Fluid Mech.* **70**, 605.

FFOWCS WILLIAMS, J. E. & LOVELY, J. 1975 *J. Fluid Mech.* **71**, 689.

FLETCHER, N. H. 1975 *J. Acoust. Soc. Am.* **57**, 233.

HOWE, M. S. 1973 *J. Sound Vib.* **27**, 455.

HOWE, M. S. 1975 *J. Fluid Mech.* **67**, 597.

KRAICHNAN, R. H. 1953 *J. Acoust. Soc. Am.* **25**, 1096.

LAMB, H. 1932 *Hydrodynamics.* Cambridge University Press.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics.* Pergamon.

LAUFER, J., FFOWCS WILLIAMS, J. E. & CHILDRESS, S. 1964 Mechanism of noise generation in a turbulent boundary layer. *AGARDograph*, no. 90.

LAUVSTAD, V. R. 1968 *J. Sound Vib.* **7**, 90.

LAUVSTAD, V. R. 1974 *J. Acoust. Soc. Am.* **56**, 1407.

LIEPMANN, H. W. & ROSHKO, A. 1957 *Elements of Gasdynamics.* Wiley.

LIGHTHILL, M. J. 1952 *Proc. Roy. Soc.* A **211**, 564.

LIGHTHILL, M. J. 1953 *Proc. Camb. Phil. Soc.* **49**, 531.

LIGHTHILL, M. J. 1954 *Proc. Roy. Soc.* A **222**, 1.

LILLEY, G. M. 1958 On noise from air jets. *Aero. Res. Counc.* 20, 376-N40-FM 2724.

LILLEY, G. M. 1973 On noise from jets. *AGARD* CP 131, paper 13.

MARBLE, F. E. 1973 Acoustic disturbance from gas nonuniformities convected through a nozzle. *Dept. Transportation Symp., Stanford.*

MORFEY, C. L. 1966 *I.S.V.R. Rep., University of Southampton.*

MORFEY, C. L. 1973 *J. Sound Vib.* **31**, 391.

MÜLLER, E. A. & MATSCHAT, K. R. 1959 The scattering of sound by a single vortex and by turbulence. *Tech. Rep. Max-Planck-Inst. für Strömungsforschung, Göttingen.*

MÜLLER, E. A. & OBERMEIER, F. 1967 The spinning vortices as a source of sound. *AGARD* CP 22, paper 22.

NAUMANN, A. & HERMANNS, E. 1973 On the interaction between a shock wave and a vortex field. *AGARD* CP131, paper 23.

OBERMEIER, F. 1967 *Acustica*, **18**, 237.

O'SHEA, J. 1971 Scattering of sound by aerodynamic flows. Ph.D. thesis, University of London.

PAO, S. P. 1971 *J. Sound Vib.* **19**, 401.

PAO, S. P. 1973 *J. Fluid Mech.* **59**, 451.

PHILLIPS, O. M. 1960 *J. Fluid Mech.* **9**, 1.

POWELL, A. 1961*a* Vortex sound. *Engng Dept., University of California, Los Angeles, Rep.* no. 61–70.

POWELL, A. 1961*b* *J. Acoust. Soc. Am.* **33**, 395.

POWELL, A. 1964 *J. Acoust. Soc. Am.* **36**, 177.

RAHMAN, S. 1971 *Acustica*, **24**, 50.

RAYLEIGH, LORD 1880 *Proc. Lond. Math. Soc.* **11**, 57.

RAYLEIGH, LORD 1945 *The Theory of Sound*. Dover.

RICHARDSON, E. G. 1931 *Proc. Phys. Soc. Lond.* **43**, 394.

SMITH, J. H. 1959 A theory of the separated flow from the curved leading edge of a slender wing. *Aero. Res. Counc. R. & M.* no. 3116.

STÜBER, B. 1970 *Acustica*, **23**, 82.

TATARSKI, V. I. 1961 *Wave Propagation in a Turbulent Medium*. McGraw-Hill.